RESEARCH NOTE

On Maxwell singularities in postglacial rebound

Lapo Boschi, Jeroen Tromp and Richard J. O’Connell
Department of Earth and Planetary Sciences, Harvard University, 20 Oxford Street, Cambridge, MA 02138, USA.
E-mail: boschi@seismology.harvard.edu

Accepted 1998 September 10; Received 1998 September 10; in original form 1998 April 29

SUMMARY
We investigate the problem of finding the numerous relaxation times associated with the postglacial rebound of a layered Maxwell earth model. In general, these relaxation times are the roots of a secular polynomial. When a numerical approach is followed, this polynomial can be very ill behaved, with a number of singularities that coincide with the Maxwell times associated with the model rheology. This problem becomes dramatically evident when the rheological profile of the model is continuous or includes a large number of uniform layers (these two cases are basically the same when the solution is computed numerically). In order to understand the physical meaning of such Maxwell singularities, we perform a comparison between the numerical approach and the existing analytical solution to the problem of the postglacial relaxation of an incompressible, self-gravitating, N-layer, spherical Maxwell earth. We show that the analytical method does not suffer from the Maxwell singularity problem, and give a theoretical explanation of the ill behaviour of the secular polynomial computed in numerical studies.

Key words: glacial rebound, mantle viscosity, normal modes, viscoelasticity.

1 INTRODUCTION
Since the seminal work of Pelitier (1974), normal-mode theory has been widely used to model the rebound of a viscoelastic earth. In this approach, the displacement field induced by a surface load is represented by a linear combination of Love numbers whose decay rates correspond to those of the relaxation modes of the Earth. As the number of layers in the model grows it becomes increasingly difficult to find these decay rates. The reason for this is that the secular polynomial, whose roots correspond to the relaxation times, has singularities at the inverse Maxwell times associated with the model rheology that may be close to its roots (Han & Wahr 1995; Fang & Hager 1994, 1995; Hanyk et al. 1996). This phenomenon is illustrated in Fig. 1 for a simple incompressible, two-layer model which has two Maxwell singularities, one associated with each layer, and two modes. The first Maxwell singularity is well isolated, but the second is close to one of the modes of the model. Each additional viscosity or rigidity discontinuity gives birth to an additional Maxwell singularity. For models with a large number of layers the numerous Maxwell singularities obscure some of the roots, which makes it numerically difficult to find all the modes.

Fang & Hager (1994, 1995) have developed a complicated contour integration technique to deal with this, in their words, 'singularity mystery'. Wu & Ni (1996) used an analytical approach to investigate very simple incompressible non-self-gravitating models by means of an algebraic manipulator. They succeeded in determining the existence of a singular factor in the secular polynomial whose contribution should be neglected when searching for the decay rates of the Earth. They suggested that their result should hold for more realistic layered models and proposed a simple way to regularize the behaviour of the secular polynomial when a numerical approach is followed.

Using elements of the semi-analytical approach to postglacial rebound modelling that characterizes the works of Sabadini et al. (1982), Spada et al. (1992), Vermeersen et al. (1996) and Vermeersen & Sabadini (1997), we show in this paper that the analytical result of Wu & Ni (1996) can be directly generalized for multiple-layer, incompressible, self-gravitating models, without the need for an algebraic manipulator. We also point out that such a semi-analytical modelling technique, which is currently employed for post-seismic rebound studies (e.g. Piersanti et al. 1995; Boschi et al. 1997), does not suffer from the 'theoretical crisis' mentioned by Fang & Hager (1995).

2 OVERVIEW OF THE THEORY
In this analysis we consider a layered, spherically symmetric, self-gravitating, incompressible, Maxwell earth model. In the
Figure 1. Example of an ill-behaved secular determinant at degree \( l = 5 \) for a two-layer earth model plotted as a function of the Laplace transform variable \( s \). Since roots of the secular polynomial are only found along the negative real axis, we restrict our interest to this portion of the complex-\( s \) plane. Singularities corresponding to the negative inverse Maxwell times are evident. We show here only a portion of the plot which includes the two Maxwell singularities associated with the two layers; other zero-crossings appear outside this region. The secular determinant plotted here is what we shall refer to as \( R_l \); we plot it in the same fashion as in Han & Wahr (1995). There, the term \( 1^+ \) was actually missing from their figures, probably due to a misprint. The model parameters are as follows, the subscript 1 denoting the upper layer and the subscript 2 denoting the lower layer: rigidity \( \mu_1 = \mu_2 = 1.45 \times 10^{11} \) Pa; density \( \rho_1 = 4120 \) kg m\(^{-3} \), \( \rho_2 = 4877 \) kg m\(^{-3} \); dynamic viscosity \( \eta_1 = 10^{20} \) Pa s, \( \eta_2 = 3 \times 10^{22} \) Pa s; the density of the fluid core is \( 10^{26} \) kg m\(^{-3} \); the radii of the outer surface, CMB and internal discontinuity are 6371, 3471 and 5701 km, respectively.

Laplace transform domain, using the correspondence principle (e.g. McConnell 1965; Cathles 1971, 1975), the response of a given homogeneous layer is governed by a system of six coupled first-order ordinary differential equations (Peltier 1974; Sabadini et al. 1984):

\[
\frac{dy}{dr} = Sy.
\]

(1)

The 6-vector \( y \) is given by

\[
y = (u, v, T_r, T_\theta, -\Phi_1, Q_1)^T,
\]

(2)

where a superscript \( T \) denotes the transpose. The six elements of the vector \( y \) depend on the Laplace transform variable \( s \), the spherical harmonic degree \( l \), and are a function of radius \( r \). The Legendre coefficients of the radial and tangential displacement are denoted by \( u \) and \( v \), respectively. The radial and tangential components of the traction have Legendre coefficients \( T_r \) and \( T_\theta \), respectively, and \( \Phi \) is the Legendre coefficient representing the incremental gravitational potential. The auxiliary variable \( Q_1 \) is defined by

\[
Q_1 = -\frac{d\Phi}{dr} - (l+1)\Phi/r + 4\pi G \rho_0 U_s,
\]

where \( \rho_0 \) denotes the unperturbed density. In each homogeneous layer, the \( 6 \times 6 \) matrix \( S \) is related to the rheology of the layer and depends on the Laplace transform variable \( s \) and the degree \( l \); an explicit expression can be found in Wu & Peltier (1982) and Sabadini et al. (1984). The \( s \) dependence of the matrix \( S \) is governed by the complex modulus \( \mu(s) = \mu(s + \mu/\eta) \), where \( \mu \) denotes the elastic shear modulus of the layer and \( \eta \) its dynamic viscosity. As we shall see, it is the \( s \) dependence of the modulus \( \mu(s) \) that causes the singularities. At the Maxwell time of the layer, determined by \( s = -\mu/\eta \), the complex modulus \( \mu(s) \) is singular. There is one such Maxwell singularity associated with each layer. The system (1) remains valid for a model that is a continuous function of radius, in which case there exists a continuous spectrum of Maxwell singularities.

Eq. (1) has been integrated both analytically and numerically in numerous studies. Its analytical solution has been determined for a homogeneous, incompressible, viscoelastic shell and can be written in the form

\[
y = Ye,
\]

(3)

where \( Y \) is a fundamental matrix that has six linearly independent solutions to eq. (1) as its columns, and \( e \) is an arbitrary 6-vector that remains to be determined by imposing the appropriate boundary conditions. An analytical expression for the fundamental matrix \( Y \) is given by e.g. Wu & Peltier (1982). Following Spada et al. (1992), we shall perform a propagation of the solution from the core–mantle boundary (CMB) at \( r = b \) to the surface of the Earth at \( r = a \), for a finite number of homogeneous layers, accounting for the boundary conditions at all interfaces.

At the CMB the solution may be written in the form

\[
y(b) = Lc,
\]

(4)

where \( b \) is the radius of the CMB, \( c \) is a constant 3-vector that remains to be determined, and the \( 6 \times 3 \) matrix \( L \), which does not depend on \( s \), is given by Sabadini et al. (1982). At each internal discontinuity the vector \( y \) is required to be continuous. The boundary conditions at the Earth's surface account for the presence of a point load (Wu & Peltier 1982). They affect only the third, fourth and sixth components of \( y \), and can be written in the form

\[
P_3 y(a) = b,
\]

(5)
where \( b \) can be found in Wu & Peltier (1982), \( a \) is the radius of our spherical model, and the projection matrix \( P_2 \) is given by
\[
P_2 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

\[ (6) \]

3 LOVE NUMBERS IN LAYERED EARTH MODELS

In the following three sections we discuss the response of one-layer, two-layer, and \( N \)-layer viscoelastic and self-gravitating earth models, governed by eq. (1) and subject to the boundary conditions (4) and (5). In each case we demonstrate analytically that the Maxwell singularities play no role in the final response.

3.1 One-layer models

Let us first consider the simplest case: an incompressible viscoelastic mantle over a fluid core. The solution to the set of first-order differential equations (1) in the Laplace domain is
\[
y(r, s) = Y(r, s)c,
\]
where \( Y \) is the fundamental matrix and \( c \) is the unknown vector that remains to be determined from the boundary conditions. At the CMB, the boundary condition (4) can be rewritten as
\[
y(b, s) = Y(b, s)c = 1\_c_c,
\]
which implies
\[
c = Y^{-1}(b, s)c_c.
\]
Substituting this expression for \( c \) into (7) and imposing the boundary condition (5) at the Earth's surface we obtain the constraint
\[
c_c = [P_2Y(a, s)Y^{-1}(b, s)c_c]^{-1}b,
\]
which determines the arbitrary 3-vector \( c_c \). Therefore,
\[
c = Y^{-1}(b, s)c_c[P_2Y(a, s)Y^{-1}(b, s)c_c]^{-1}b,
\]
and the solution (7) takes the form
\[
y(r, s) = Y(r, s)Y^{-1}(b, s)c_c[P_2Y(a, s)Y^{-1}(b, s)c_c]^{-1}b.
\]
Since we are interested in the value of the displacement and gravitational perturbation at the Earth's surface, represented by the first, second and fifth elements of \( y \), we define a second projection matrix,
\[
P_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}.
\]
\[ (13) \]

such that the three components mentioned above are contained in the 3-vector
\[
x(a, s) = P_1y(a, s).
\]
\[ (14) \]

In order to obtain a more compact notation, we define the \( 3 \times 3 \) matrices
\[
R = P_2Y(a, s)Y^{-1}(b, s)c_c,
\]
\[ (15) \]
\[
Q = P_2Y(a, s)Y^{-1}(b, s)c_c,
\]
\[ (16) \]

such that we can write
\[
x(a, s) = QR^{-1}b = Q\frac{R^t}{|R|}b,
\]
\[ (17) \]

where \( R^t \) denotes the adjoint of \( R \) and \( |R| \) its determinant.

Both \( Y(r, s) \) and \( Y^{-1}(r, s) \) have been determined analytically by Spada et al. (1992). Their expressions contain some misprints that are reported and corrected in Vermeersen et al. (1996). It is possible to define four \( 6 \times 6 \) matrices \( A, B, C, D \), independent of \( s \), such that \( Y(a, s) \) and \( Y^{-1}(b, s) \) can be written in the form
\[
Y(a, s) = A + \mu(s)B,
\]
\[ (18) \]
\[
Y^{-1}(b, s) = C + \frac{1}{\mu(s)}D.
\]
\[ (19) \]

After some algebra, we find that
\[
Y(a, s)Y^{-1}(b, s) = [s(s + \mu/\eta)]^{-1}[s(s + \mu/\eta)(AC + BD) + \mu s^2BC + \mu^{-1}(s + \mu/\eta)^2AD].
\]
\[ (20) \]

The factor \( [s(s + \mu/\eta)]^{-1} \) is clearly singular at \( s = 0 \) and \( s = -\mu/\eta \), but the \( 6 \times 6 \) matrix between the square brackets is a regular function of \( s \). For the sake of simplicity, we denote it as
\[
N(s) = s(s + \mu/\eta)(AC + BD) + \mu s^2BC + \mu^{-1}(s + \mu/\eta)^2AD.
\]
\[ (21) \]

It then follows that
\[
R = [s(s + \mu/\eta)]^{-1}P_2N_c,
\]
\[ (22) \]
\[
Q = [s(s + \mu/\eta)]^{-1}P_2N_c,
\]
\[ (23) \]
\[
|R| = [s(s + \mu/\eta)]^{-3}[P_2N_c]'
\]
\[ (24) \]

and
\[
R^t = [s(s + \mu/\eta)]^{-2}[P_2N_c]'.
\]
\[ (25) \]

Therefore, the solution (17) may be rewritten in the form
\[
x(a, s) = [s(s + \mu/\eta)]^{-1}P_1N_c\frac{[s(s + \mu/\eta)]^{-1}[P_2N_c]'}{[s(s + \mu/\eta)]^{-3}[P_2N_c]'}b
\]
\[
= P_1N_c\frac{[P_2N_c]'}{[P_2N_c]'}b,
\]
\[ (26) \]

where the singular factor \( [s(s + \mu/\eta)]^{-1} \) cancels out. The inverse Laplace transform of the Love number vector \( x(a, s) \) is simply a combination of terms that decay exponentially with respect to time, and each decay rate coincides with a singularity of \( x(a, s) \) in the Laplace domain. Since \( P_2N_c \) and \( P_1N_c \) are regular functions of \( s \), it is clear that the only singularities of \( x(a, s) \) in the one-layer case are the roots of \( [P_2N_c]' \), a well-behaved function of \( s \). These roots need to be determined numerically, and this is why we call the approach presented here semi-analytical rather than analytical.

As shown explicitly in Section 4, in numerical studies the viscoelastic modes of the Earth are determined by finding the roots of the numerical analogues of \( |R| \), Fig. 2 illustrates for a simple one-layer model the difference between monitoring the determinant \( |P_2N_c| \) and the ill-behaved determinant \( |R| \); the Maxwell singularity associated with the latter determinant is evident. The zero crossing in \( |P_2N_c| \) at the same location
where \( |R| \) is singular is associated with what we will refer to as a 'Maxwell mode'; this mode does not carry any strength (Spada et al. 1992; Vermeersen & Sabadini 1997).

### 3.2 Two-layer models

Models with more than one layer must be handled in a slightly different fashion. We have to propagate the solution from one layer to another by taking into account the condition of continuity of \( y \), which still has the general form (3) within each layer, at each interface. We will now show in detail how this is done for a two-layer model in order to generalize our result to the case of an arbitrary number of layers in the next section. Let \( a \) be the Earth's radius, \( d \) the radius of the discontinuity between the upper and lower mantle, and \( b \) the radius of the CMB. We indicate the upper mantle and its parameters with a subscript 1, while a subscript 2 denotes the lower mantle. The boundary condition at the CMB (4) implies

\[
Y_2(b, s) c_2 \equiv c_2 = Y_2^{-1}(b, s) b.
\]

The condition of continuity at the boundary \( r = d \) between the two layers implies

\[
y_1(d, s) = y_2(d, s) \Rightarrow Y_1(d, s) c_1 = Y_2(d, s) c_2 \Rightarrow c_2 = Y_1^{-1}(d, s) Y_2(d, s) Y_2^{-1}(b, s) b.
\]

If we now impose the boundary conditions (5) at the Earth's surface, we obtain

\[
c_0 = [P_2 Y_1(a, s) Y_1^{-1}(d, s) Y_2(d, s) Y_2^{-1}(b, s) b]^{-1} b,
\]

which is substituted into (28) and subsequently into the expression for the solution vector at the surface \( y_1(a, s) = Y_1(a, s) c_1 \). As a result, \( y_1(a, s) \) is now uniquely determined. By analogy with the one-layer case, we define the 3 \times 3 matrices

\[
R = P_2 Y_1(a, s) Y_1^{-1}(d, s) Y_2(d, s) Y_2^{-1}(b, s) b,
\]

\[
Q = P_1 Y_1(a, s) Y_1^{-1}(d, s) Y_2(d, s) Y_2^{-1}(b, s) b.
\]

For simplicity, we use the same notation as in the previous section even though \( R \) and \( Q \) are now different. As in the previous section, we can write the Love number vector in the compact form

\[
x(a, s) = Q R |R|^{-1} b.
\]

Rewriting the matrices \( Y_i \) and \( Y_i^{-1} \), \( i = 1, 2 \), in a similar fashion as for the one-layer case,

\[
Y_i(r_i, s) = A_i + \mu_i(s) B_i,
\]

\[
Y_i^{-1}(r_i, s) = C_i + \frac{1}{\mu_i(s)} D_i,
\]

where \( r_1 = a \), \( r_2 = d \), \( r_3 = b \) and \( \mu_i(s) = \mu_i(s)/(s + \mu_i/\eta_i) \), \( i = 1, 2 \), we find after some algebra the identity

\[
Y_1(a, s) Y_1^{-1}(d, s) Y_2(d, s) Y_2^{-1}(b, s)
\]

\[
= [(s + \mu_1/\eta_1)(s + \mu_2/\eta_2) s^2]^{-1} N(s),
\]

where the 6 \times 6 matrix

\[
N(s) = \prod_{j=1}^{2} [s(s + \mu_j/\eta_j)(A_j C_j + B_j D_j) + \mu_j s^2 B_j C_j + \mu_j (s + \mu_j/\eta_j)^2 A_j D_j]
\]

is a regular function of \( s \). It is now clear that

\[
R = [(s + \mu_1/\eta_1)(s + \mu_2/\eta_2) s^2]^{-1} P_2 N(s).
\]
and, likewise,
\[ Q = \left[ (s + \mu_1/\eta_1)(s + \mu_2/\eta_2)s^2 \right]^{-1} P_1 NL_e. \] (38)
Moreover,
\[ |R| = \left[ (s + \mu_1/\eta_1)(s + \mu_2/\eta_2)s^2 \right]^{-3} |P_2 NL_e|. \] (39)
\[ R' = \left[ (s + \mu_1/\eta_1)(s + \mu_2/\eta_2)s^2 \right]^{-2} \left[ P_2 NL_e \right]. \] (40)
Eq. (32) can now be written as
\[ x(a, s) = P_1 NL_e \left[ P_2 NL_e \right]^T b. \] (41)
Again, the modes are roots of the secular determinant \(|P_2 NL_e|\), which is a regular function of \(s\). In Fig. 3, \(|P_2 NL_e|\) and \(|R|\) are plotted as functions of \(s\) for a two-layer model, focusing on the vicinity of one of the two Maxwell singularities, \(s = -\mu_i/\eta_i\). For this model, as in the one-layer case, Maxwell singularities do not pose a problem, since all the roots of \(|P_2 NL_e|\) are also roots of \(|R|\); on the other hand, the singularities in \(|R|\) correspond to zeroes in \(|P_2 NL_e|\), but, as in the one-layer case, these roots do not carry any strength.

3.3 \(N\)-layer models

The results in the previous sections are readily generalized to earth models with multiple layers. To streamline the notation, we denote the radius of the Earth by \(r_1\), the radii of all internal discontinuities by \(r_i\), \(i = 2, \ldots, N\), and the radius of the CMB by \(r_{N+1}\). In general, a subscript \(i\) labels the layer bounded by the interfaces with radii \(r_i\) and \(r_{i+1}\).

Again, we start from the condition (4) at the CMB,
\[ Y_N(r_{N+1}, s) c_N = I_c c_e \Rightarrow c_N = Y_N^{-1}(r_{N+1}, s) I_c c_e. \] (42)
Using the same propagation technique employed above, iterated for \(N\) layers, we write the solution vector at the surface in terms of the arbitrary 3-vector \(c_e\):
\[ y_1(r_1, s) = \left\{ \prod_{i=1}^{N} \left[ Y_i(r_i, s) Y_i^{-1}(r_{i+1}, s) \right] \right\} I_c c_e. \] (43)
We then impose upon (43) the boundary conditions at the Earth's surface, thus determining the arbitrary constant \(c_e\):
\[ c_e = \left\{ \prod_{i=1}^{N} \left[ Y_i(r_i, s) Y_i^{-1}(r_{i+1}, s) \right] \right\}^{-1} b. \] (44)
Expression (44) for \(c_e\) can then be substituted into (43). At this point the notation is compacted by defining the matrices \(R\) and \(Q\) for the \(N\)-layer case, isolating the singular factor in exactly the same way as above:
\[ R = \left\{ \prod_{i=1}^{N} \left[ s(s + \mu_i/\eta_i) \right]^{-1} \right\} P_2 NL_e, \] (45)
\[ Q = \left\{ \prod_{i=1}^{N} \left[ s(s + \mu_i/\eta_i) \right]^{-1} \right\} P_1 NL_e, \] (46)
where
\[ N = \left\{ \prod_{i=1}^{N} \left[ s(s + \mu_i/\eta_i) \right] \right\} \left[ (A, D_i) + \mu_i s \left[ B_i C_j + \mu_j^{-1} (s + \mu_j/\eta_j)^2 A_j D_j \right] \right]. \] (47)

Now the vector \(x(r_1, s) = P_1 Y_1(r_1, s)\) has, once again, the form (17). If expressions (45) and (46) for \(R\) and \(Q\) are subsequently substituted into (17), it can be verified that the secular singular factor \(\prod_{i=1}^{N} \left[ s(s + \mu_i/\eta_i) \right]^{-1}\) cancels out, as in the previous examples, and that the secular determinant is a regular function of \(s\).

4 DISCUSSION AND CONCLUSIONS

We have presented a semi-analytical approach to the problem of finding the relaxation modes of a layered, incompressible earth model. In particular, we have shown that if this technique...
that we can use $R$ to explain the behavior of its numerical analogue $M$ and the errors that numerical techniques might introduce (Han & Wahr 1995; Fang & Hager 1994, 1995). Specifically, in the previous sections we have determined that the matrix $R$ includes the factor $\{\lambda_{s}^{-1} + \mu_s / \eta_s\}^{-1}$, which is singular at $s = -\mu_s / \eta_s$. i.e. when the Laplace variable $s$ equals the negative inverse Maxwell time of a layer. This is exactly the 'singular factor' also noticed by Wu & Ni (1996) for very simple non-self-gravitating models. If an analytical approach is used, we do not need to monitor $R(a)$ as a function of $s$ in order to find the relaxation times; indeed, our algebraic manipulation of the analytical expression for the Love number vector $x(a, s)$ has shown that the singular factor cancels out and does not appear in the final 'corrected' expression for the secular polynomial $[P_{2N}l]$, which is a regular function of $s$.

Numerical algorithms, on the other hand, are forced to find the roots of the ill-behaved function $R(a, s)$, using its numerical analogue $M(a, s)$; Fang & Hager (1995) and Han & Wahr (1995) confirm that in their numerical calculations the negative inverse Maxwell times of the model turn out to be singularities of the secular polynomial, and that such singularities greatly complicate the determination of the true relaxation modes of the Earth when a large number of layers is involved. This problem exists for both compressible and incompressible models (James 1991; Han & Wahr 1995).

Wu & Ni (1996), based upon their analysis of simplified non-self-gravitating models with a limited number of layers, suggested a way to eliminate the Maxwell singularities that always arise in numerical rebound calculations: the numerical secular determinant can be 'regularized' by simply multiplying it by the inverse of the singular factor [see the discussion of eq. (14) in Wu & Ni (1996)]. The main achievement of our work is a rigorous proof, for very general N-layer and self-gravitating incompressible models, of the speculation of Wu & Ni (1996). Even though we have discussed a way to improve numerical techniques, we also point out that analytical solutions are available for layered, incompressible, self-gravitating Maxwell earth models with an arbitrary number of layers (Spada et al. 1992; Vermeersen et al. 1996; Vermeersen & Sabadini 1997). Maxwell singularities are not an issue when the analytical approach is used.

ACKNOWLEDGMENTS

We are grateful to Giorgio Spada for his suggestions and support. We would like to thank Gaia Soldati, Antonio Piersanti and an anonymous reviewer for their useful comments.

REFERENCES


© 1999 RAS, *GJI* 136, 492–498