Stationary-phase integrals in the cross-correlation of ambient noise

Lapo Boschi\textsuperscript{1,2} and Cornelis Weemstra\textsuperscript{3}

\textsuperscript{1}Sorbonne Universités, UPMC Univ Paris 06, UMR 7193, Institut des Sciences de la Terre Paris (iSTeP), F-75005 Paris, France.
\textsuperscript{2}CNRS, UMR 7193, Institut des Sciences de la Terre Paris (iSTeP), F-75005 Paris, France.
\textsuperscript{3}Department of Geoscience and Engineering, Delft University of Technology, Stevinweg 1, 2628 CN Delft, The Netherlands.

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Abstract

The cross-correlation of ambient signal allows seismologists to collect data even in the absence of seismic events. “Seismic interferometry” shows that the cross-correlation of simultaneous recordings of a random wave field made at two locations is formally related to the impulse response between those locations. This idea has found many applications in seismology, as a growing number of dense seismic networks become available: cross-correlating long seismic records, the Green’s function between instrument pairs is “reconstructed,” and used, just like the seismic recording of an explosion, in tomography, monitoring, etc. These applications have been accompanied by theoretical investigations of the relationship between noise cross-correlation and the Green’s function; numerous formulations of “ambient-noise” theory have emerged, each based on different hypotheses and/or analytical approaches. The purpose of this study is to present most of those approaches together, providing a comprehensive overview of the theory. Understanding the specific hypotheses behind each Green’s function recipe is critical to its correct application. Hoping to guide non-specialists who approach ambient-noise theory for the first time, we treat the simplest formulation (the stationary-phase approximation applied to smooth unbounded media) in detail. We then move on to more general treatments, illustrating that the “stationary-phase” and “reciprocity-theorem” approaches lead to the same formulae when applied to the same scenario. We show that a formal cross-correlation/Green’s function relationship can be found in complex, bounded media, and for nonuniform source distributions. We finally provide the bases for understanding how the Green’s function is reconstructed in the presence of scattering obstacles.

1 Introduction

In “seismic interferometry,” the Green’s function, or impulse response, of a medium can be determined empirically, based on the background signal recorded by two instruments over some time. The term “Green’s function”, ubiquitous in ambient-noise literature, indicates the response of a medium to an impulsive excitation: a point source [e.g., Morse and Ingard, 1986; Aki and Richards, 2002]. Measuring a Green’s function amounts to recording the ground oscillations that follow an explosion: exploiting the “ambient noise,” the same signal can be measured without setting off any explosive. This approach was foreshadowed by several early studies in ocean acoustics [e.g., Eckart, 1953; Cox, 1973] and small-scale seismology [Aki, 1957; Claerbout, 1968], and later applied successfully to helioseismology [Duvall et al.,
Ambient-noise seismology on Earth takes advantage of the “ambient”, low-energy signal that seems to be generated continuously by the coupling between oceans and solid Earth [e.g., Longuet-Higgins, 1950; Hasselmann, 1963; Stehly et al., 2006; Kedar et al., 2008; Hillers et al., 2012; Gualtieri et al., 2013; Traer and Gerstoft, 2014]. Its resolution is not limited by the nonuniform distribution of earthquakes, or by the difficulties inherent in setting up a man-made seismic source. Most ambient energy pertains to surface waves of period between ~5s and ~30s, which are difficult to observe in the near field of an earthquake, and almost completely attenuated in the far field, where only longer-period surface waves can be measured. Ambient-noise-based surface-wave observations in this frequency range are thus complementary to traditional observations, and particularly useful to map the crust, lithosphere and asthenosphere: the shorter the surface-wave period, the shallower the depth range. Some recent studies [Ruigrok et al., 2011; Ito and Shiomi, 2012; Poli et al., 2012a, b; Gorbatov et al., 2013; Nishida, 2013] show that not only surface-waves, but also body waves traveling over large distances and/or reflected by Earth discontinuities (the Moho, the upper-to-lower mantle boundary, etc.) can be extracted from ambient-signal cross-correlations.

Ambient-noise correlation allows one to build a seismic data set even in the absence of earthquakes. In practice, the Green’s function is reconstructed only approximately and within a limited frequency band, but that is enough to estimate relevant parameters such as group and phase velocity of surface waves. This is valuable for a number of different applications, allowing passive imaging at the reservoir scale and in the context of hydrocarbon industry [De Ridder and Dellinger, 2011; Corciulo et al., 2012; Weemstra et al., 2013] as well as monitoring of time-dependent changes in material properties around an active fault [Wegler and Sens-Schonfelder, 2007], in the rigidity of a landslide-prone area [Mainsant et al., 2012], or in the shape and location of magma [Brenguier et al., 2011] and hydrocarbon [De Ridder et al., 2014] reservoirs.

The last decade has seen the publication of a number of increasingly exhaustive mathematical descriptions of the phenomenon of Green’s function reconstruction from ambient recordings [Lobkis and Weaver, 2001; Derode et al., 2003; Snieder, 2004; Wapenaar, 2004; Weaver and Lobkis, 2004; Roux et al., 2005; Wapenaar et al., 2005; Nakahara, 2006; Sanchez-Sesma and Campillo, 2006; Wapenaar et al., 2006; Tsai, 2009; Weaver et al., 2009].
general, a mathematical relationship is found between the Green’s function associated with
the locations of two receivers (i.e., the response, observed at one of the receivers, to a point
source deployed at the other receiver) and the cross-correlation [e.g. Smith, 2011], computed
over a long time interval, between the random ambient signal recorded by the receivers. (A
detailed account of how ambient data are treated is given by Bensen et al. [2007].) Different
approaches to establishing this relationship have been followed, however, resulting in
different formulations. Most authors first develop the theory for the simple case of acoustic
(scalar) wave propagation in two- or three-dimensional media. Some formulations hold for
heterogeneous and/or bounded media, while others are limited to infinite homogeneous me-
dia. Defining ambient noise (a “diffuse field”) mathematically is a nontrivial problem per se:
some authors describe it as the superposition of plane waves propagating in all directions;
others prefer to superpose impulsive responses (Green’s functions), sometimes in two dimen-
sions, sometimes in three; others yet define the noise field as one where all normal modes
have the same probability of being excited. Finally, the expression “Green’s function” itself
is ambiguous: in elastodynamics, it may refer to the response of a medium to an impulsive
force, impulsive stress, or impulsive initial conditions in displacement, or velocity... While
specialists are well aware of these differences and their implications, interferometry “users”
(including the authors of this article) are sometimes confused as to the theoretical basis, and
practical reliability, of the methods they apply.

The goal of this study is to show in detail how several different derivations of “ambient-
noise theory” lead to apparently different, but indeed perfectly coherent results. We first
treat some particularly simple scenarios: ambient noise in a homogeneous lossless (i.e. non-
attenuating) 2-D or 3-D medium, generated by azimuthally uniform, 2-D or 3-D distributions
of point sources, or by plane waves propagating in all directions. The analogy between acous-
tic waves in two dimensions and Rayleigh waves is discussed. Analytical expressions for the
cross-correlation of ambient signals in all these physical settings are given in sec. 3 and im-
plemented numerically in sec. 4. In all the simple cases we consider, the cross-correlation
of ambient signal as derived in sec. 3 coincides with the integral, over the area occupied by
sources, of an overall very oscillatory function, which becomes slowly varying only around a
small set of so-called “stationary” points. We show in sec. 5 how such integrals are solved via
the approximate “stationary-phase” method (appendix A); different analytical relationships
between cross-correlation and Green’s function are thus found and discussed; they are later
summarized in sec. 9. In sec. 8 we apply, again, the stationary-phase method to a more
complicated medium including one scattering obstacle, which allows us to introduce the
concept, often found in ambient-noise literature, of “spurious” arrival in cross-correlation.
Finally, the so-called “reciprocity-theorem” approach provides an analytical relationship be-
tween Green’s function and cross-correlation that is valid in arbitrarily complex, attenuating
media. We describe it in detail in sec. 6, together with a few other “alternative” approaches
to ambient-noise theory; the consistency between reciprocity-theorem and stationary-phase
results is verified. To avoid ambiguity as much as possible, we provide an overview of the
underlying theory for both acoustics and elastodynamics in sec. 2, and a derivation of acous-
tic Green’s functions in appendix E. By collecting all this previously scattered material in
a single review, we hope to provide a useful tool for graduate students and non-specialists
approaching the theory of acoustic and seismic ambient noise for the first time. Most of the
results presented here are limited to non-scattering, non-dissipative, homogeneous acoustic
media, and to surface-wave (and not body-wave) propagation in elastic media. These sim-
plifications have the advantage of allowing a self-contained, relatively uncluttered derivation.
We explore more realistic setups (non-uniform source distributions; scattering) in secs. 7 and
8. The thorough understanding of ambient-noise cross-correlation in simple media will serve
as a solid platform for more advanced investigations. Readers that are mostly interested in
applications to realistic environments are referred to the differently minded reviews of Curtis
et al. [2006], Larose et al. [2006], Gouedard et al. [2008], Wapenaar et al. [2010a, b], Snieder
and Larose [2013], Ritzwoller [2014] and Campillo and Roux [2014].

2 Governing equations

In this study, analytical relationships between Green’s functions and the cross-correlation
of ambient signal are derived in a number of different scenarios: spherical acoustic waves
in two and three dimensions, Rayleigh waves, plane waves. We first summarize the theory
underlying each of these cases.

2.1 Acoustic waves from a point source in free space

Pressure $p$ in homogeneous, inviscid fluids occupying a three-dimensional space obeys the
linear, lossless wave equation

$$\nabla^2 p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = \frac{\partial q}{\partial t}, \quad (1)$$
where $\nabla$ is the gradient operator, $c^2$ the ratio of adiabatic bulk modulus to density, $t$ denotes time and the forcing term $q$ is the (apparent) mass production per unit volume per unit time, representing e.g. an explosion or a loudspeaker [e.g., Kinsler et al., 1999, chapter 5].

If $\partial q/\partial t = \delta(r - r_0)\delta(t - t_0)$, with $\delta$ the Dirac distribution, $r$ the position vector and $r_0$, $t_0$ the location and time of an impulsive sound source, respectively, eq. (1) is referred to as “Green’s problem”, and its solution for $p$ as “Green’s function”. Once the Green’s problem is solved, the response of the system to however complicated a source is found by convolving the Green’s and source functions [e.g., Morse and Ingard, 1986; Aki and Richards, 2002].

The Green’s function $G_{3D}$ associated with (1) is derived in appendix E here, eqs. (200) and (201), working in spherical coordinates and choosing (without loss of generality) $t_0 = 0$ and $r_0 = 0$.

2.2 Membrane waves from a point source

Eq. (1) also describes the motion of an elastic membrane. In this case, $p$ can be interpreted as displacement in the vertical direction (for an horizontal membrane), $\nabla$ is the surface gradient and $c$ the ratio of the membrane tension per unit length to its surface density [e.g., Kinsler et al., 1999, section 4.2]. The associated Green’s function $G_{2D}$ is determined analytically in appendix E, eqs. (194) and (195).

2.3 Rayleigh waves from a point source

The “potential” representation of surface-wave propagation, adopted e.g. by Tanimoto [1990], consists of writing Rayleigh-wave displacement

$$u_R = U(z) \hat{z} \Psi_R(x,y,t) + V(z) \nabla_1 \Psi_R(x,y,t),$$

where $x$, $y$, $z$ are Cartesian coordinates ($z$ denotes depth), the “vertical eigenfunctions” $U$ and $V$ depend on $z$ only and the Rayleigh wave “potential” $\Psi_R$ varies only laterally (but varies with time). $\hat{x}$, $\hat{y}$, $\hat{z}$ are unit vectors in the directions of the corresponding coordinates, and the operator $\nabla_1 = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y}$. Love-wave displacement is accordingly written

$$u_L = -W(z) \hat{z} \times \nabla_1 \Psi_L(x,y,t),$$

with $\times$ denoting a vector product.

Substituting the Ansätze (2) and (3) into the 3-D equation of motion, one finds that both
potentials $\Psi_R$ and $\Psi_L$ satisfy

$$\nabla^2 \Psi_{R,L} - \frac{1}{c_{R,L}^2} \frac{\partial^2 \Psi_{R,L}}{\partial t^2} = T_{R,L},$$

i.e. eq. (1), with $c_{R,L}$ denoting phase velocity, and $T_R, T_L$ scalar forcing terms [e.g. Tanimoto, 1990; Tromp and Dahlen, 1993; Udías, 1999].

The Green’s problem associated with (4) coincides with the 2-D, membrane-wave problem of sec. 2.2 and appendix E, and the Green’s function corresponding to $\Psi_{R,L}$ is, again, $G_{2D}$ of eqs. (194) and (195). The surface-wave Green’s function can be obtained from the scalar Green’s function $G_{2D}$ via eq. (2) [Tromp and Dahlen, 1993, sec. 5].

Importantly, according to (2) the vertical component of Rayleigh-wave displacement $u_R$ is simply $U(z)\Psi_R(x,y;t); U(z)$ is a function of $z$ that does not change with time, so that the phase of the vertical component of $u_R$ coincides with that of the potential $\Psi_R(x,y;t)$. The analysis of 2-D ambient noise that we conduct here holds therefore for Rayleigh waves propagating in a 3-D medium (in the Earth) and measured on the vertical component of seismograms, as far as the phase is concerned; amplitude needs to be scaled.

The Rayleigh-wave derivation of Snieder [2004] is also based on this idea, although in that study the separation between $x,y,t$-dependence and $z$-dependence follows from a normal-mode formulation [e.g., Snieder, 1986]. Halliday and Curtis [2008] explore the effects of sub-surface sources, or lack thereof, on Green’s function reconstruction; their results are consistent with ours and with those of Snieder [2004] whenever their setup is the same as ours, i.e. uniform source distribution over the Earth’s (laterally invariant) surface.

### 2.4 Plane waves

Some authors [e.g., Aki, 1957; Sanchez-Sesma and Campillo, 2006; Tsai, 2009; Boschi et al., 2013] have studied diffuse wave fields with cylindrical symmetry, which they described as the superposition of plane waves (rather than spherical or circular as seen so far) traveling along many different azimuths. If its right-hand side $\frac{\partial q}{\partial t}=0$, eq. (1) is satisfied by a monochromatic plane wave

$$p(\mathbf{r},t) = S(\omega) \cos(\omega t + \mathbf{k} \cdot \mathbf{r}),$$

where the vector $\mathbf{k}$ is constant and defines the direction of propagation, and the amplitude function $S(\omega)$ is arbitrary. A plane wave approximates well the response of a medium to a real source, at receivers that are sufficiently far from the source.
3 Cross-correlation in a diffuse wave field

We are interested in the cross-correlation between recordings of a diffuse wave field made at a pair of receivers. In acoustics, a diffuse wave field is such that the energy associated with propagating waves is the same at all azimuths of propagation [e.g. Kinsler et al., 1999, sec. 12.1]. (The expression “equipartitioned field”, often intended as a synonym of “diffuse”, is used in the literature with slightly different meanings depending on the context [Snieder et al., 2010], and we chose not to employ it here to avoid ambiguity.) While ambient noise recorded on Earth is not strictly diffuse [Mulargia, 2012], diffuse-field theory successfully describes many seismic observations, and as such it is at least a useful first approximation of real ambient noise.

We simulate an approximately diffuse wave field by averaging (or, in seismology jargon, “stacking”) cross-correlations associated with sources distributed uniformly over a circle or a sphere surrounding the receivers. This is equivalent to the source azimuth being random, i.e., over time, all azimuths are sampled with equal frequency/probability.

We center our spherical reference frame at the location of “receiver 1” (R1), and orient it so that “receiver 2” (R2) lies on the $\varphi = 0, \theta = \pi/2$ axis (Fig. 1). We choose sources to be separated in time, that is to say, a receiver never records signal from more than one
source at the same time: while diffuse wave fields in the real world might result from multiple, simultaneous sources, our simplification is justified by the mathematical finding that the so-called “cross-terms”, i.e. the receiver-receiver cross-correlation of signal generated by different sources, are negligible when “ensemble-averaged”. A proof is given e.g. by Weemstra et al. [2014] and is summarized here in appendix D.

Let us call $S$ a source location, and $r_{iS}$ the distance between $S$ and the $i$-th receiver. If the source at $S$ is an impulsive point source, those signals coincide with the Green’s function associated with the source location $S$, evaluated at $R1$ and $R2$. In 3-D, the Green’s function is given by eq. (201), and in the frequency domain the cross-correlation reads

$$G_{3D}(r_{1S}, \omega) G^\ast_{3D}(r_{2S}, \omega) = \frac{1}{(4\pi c)^2} \frac{1}{2\pi} \frac{e^{i\frac{c}{r_{1S}}(r_{2S}-r_{1S})}}{r_{1S} r_{2S}}$$  \hspace{1cm} (6)

(here and in the following the superscript $\ast$ denotes complex conjugation). Since the dimension of $G_{3D}$ in the frequency domain is time over squared distance (app. E), that of eq. (6) is squared time over distance to the power of four, or time over distance to the power of four in the time domain.

In 2-D (membrane waves, Rayleigh waves), $G_{2D}$ is given by (195), and the cross-correlation

$$G_{2D}(r_{1S}, \omega) G^\ast_{2D}(r_{2S}, \omega) = -\frac{i}{32\pi c^4} H_0^{(2)} \left( \frac{\omega r_{1S}}{c} \right) \left[ -i H_0^{(2)} \left( \frac{\omega r_{2S}}{c} \right) \right]^\ast.$$  \hspace{1cm} (7)

The cumulative effect of multiple sources $S$ is obtained by rewriting expressions (6) or (7) for each $S$, with $r_{1S}$ and $r_{2S}$ varying as functions of the distance $r$ and azimuth $\varphi$ (and, in 3-D, inclination $\theta$) of $S$ with respect to the origin, and summing.

We shall consider several different scenarios:

(i) In 3-D space, for a continuous distribution of sources along a circle in the $\theta = \pi/2$ plane (centered, for the sake of simplicity, at $R1$) summing the cross-correlations over $S$ leads to an integral over $\varphi$,

$$I_C(\omega) = \frac{1}{(4\pi c)^2} \frac{1}{4\pi^2} \int_{-\pi}^{\pi} d\varphi \ n_C(\varphi) \frac{e^{i\frac{c}{r_{1S}}(r_{2S}(\varphi)-r_{1S}(\varphi))}}{r_{1S}(\varphi) r_{2S}(\varphi)},$$  \hspace{1cm} (8)

with $n_C$ the number of sources per unit azimuth, or source density.

(ii) If sources are distributed on a sphere centered at $R1$, the double integral is over the
surface of the sphere (4π solid radians), and

\[ I_S(\omega) = \frac{1}{4\pi} \frac{1}{(4\pi c)^2} \frac{1}{2\pi} \int_0^\pi d\theta \sin \theta \int_{-\pi}^{\pi} d\varphi \, n_S(\theta, \varphi) G_{3D}(r_{1S}, \omega) G_{3D}^*(r_{2S}, \omega) \]

\[ = \frac{1}{4\pi} \frac{1}{(4\pi c)^2} \frac{1}{2\pi} \int_0^\pi d\theta \sin \theta \int_{-\pi}^{\pi} d\varphi \, n_S(\theta, \varphi) \frac{e^{i \frac{\omega}{c} (r_{2S}(\theta, \varphi) - r_{1S}(\theta, \varphi))}}{r_{1S}(\theta, \varphi) r_{2S}(\theta, \varphi)}, \tag{9} \]

where \( n_S \) denotes the number of sources per unit of solid angle on the sphere.

(iii) Switching to 2-D, i.e. to Rayleigh waves or elastic waves propagating on a membrane, we need to integrate expression (7) over the position occupied by \( S \). Assuming a source distribution analogous to that of case (i), with sources along a circle, and source density \( n_M \) a function of \( \varphi \) only, we find after some algebra that cross correlation is described by

\[ I_{MW}(\omega) = \frac{1}{32\pi c^4} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi \, n_M(\varphi) H_0^{(2)} \left( \frac{\omega r_{1S}(\varphi)}{c} \right) \frac{1}{\omega} \frac{H_0^{(2)*} \left( \frac{\omega r_{2S}(\varphi)}{c} \right)}{H_0^{(2)} \left( \frac{\omega r_{2S}(\varphi)}{c} \right)} \]

\[ \approx \frac{1}{16\pi^2 c^3} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi \, n_M(\varphi) \frac{e^{i \frac{\omega}{c} (r_{1S}(\varphi) - r_{2S}(\varphi))}}{\omega \sqrt{r_{1S}(\varphi) r_{2S}(\varphi)}}, \tag{10} \]

where we have replaced the Hankel function \( H_0^{(2)} \) with its asymptotic (high-frequency and/or far-field) approximation, eq. (9.2.4) of Abramowitz and Stegun [1964]; this approximation is necessary to later solve the integral in (10) via the stationary-phase method. Notice that we do not yet require the wave field to be isotropic: at this point \( n_C, n_S, n_M \) are arbitrary functions of \( \theta, \varphi \). We will show in sec. 5, however, that they need to be smooth or constant for the stationary-phase approximation to be applicable.

(iv) In the plane-wave approach of sec. 2.4 no source location is specified, but one combines plane waves (5) traveling along all azimuths \( \varphi \). At \( R1 \), the monochromatic plane wave of frequency \( \omega_0 \) traveling in the direction \( \varphi \) is

\[ p(R1, t) = S(\omega_0) \cos(\omega_0 t); \tag{11} \]

at \( R2 \) the same signal is recorded at a different time, and

\[ p(R2, t) = S(\omega_0) \cos \left[ \omega_0 \left( t + \frac{\Delta \cos \varphi}{c} \right) \right]. \tag{12} \]

Based on eqs. (11) and (12), Boschi et al. [2013] show that the source-averaged, two-
station cross-correlation of monochromatic plane waves at frequency $\omega_0$ can be written

$$I_{PW}(\omega_0, t) = \frac{q(\omega_0)}{2\pi} \int_0^\pi d\varphi \cos \left[ \omega_0 \left( t + \frac{\Delta \cos \varphi}{c} \right) \right],$$

(13)

obtained from eqs. (18) and (21) of Boschi et al. [2013] through the change of variable $t_d = \Delta \cos(\varphi)/c$, and assuming source density and the amplitude term $q(\omega)$ (reflecting the frequency spectrum of the source) to be constant with respect to $\varphi$. The Fourier transform of (13) is

$$I_{PW}(\omega_0, \omega) = \frac{q(\omega_0)}{\sqrt{8\pi}} \int_0^\pi d\varphi \left[ \delta(\omega + \omega_0) + \delta(\omega - \omega_0) \right] e^{i\omega \Delta \cos \varphi}. \quad \text{(14)}$$

Eq. (14) is valid for any $\omega_0$; using the properties of the Dirac $\delta$ function we generalize it to

$$I_{PW}(\omega) = \frac{q(\omega)}{\sqrt{2\pi}} \int_0^\pi d\varphi \cos \left( \omega \frac{\Delta \cos \varphi}{c} \right). \quad \text{(15)}$$

We integrate eqs. (8)-(10) and (15) numerically in sec. 4; this exercise will serve to validate the results, illustrated in sec. 5, of integrating the same equations analytically via the stationary-phase approximation.

The case of a spatially (and not just azimuthally) uniform distribution of sources over a plane or the 3-D space is not treated here in the interest of brevity. It essentially requires that expressions (8), (9) or (10) be additionally integrated over source location; in practice, one needs to integrate over the distance between source and origin, along the azimuths of stationary points only (one-dimensional integrals) [Snieder, 2004; Sato, 2010].

### 4 Numerical integration

Before deriving approximate analytical solutions for the integrals in eqs. (8)-(10) and (15), we implement these equations numerically. In practice, we evaluate eqs. (195) and (201) at $R1, R2$, to determine of $G_{2D}$ or $G_{3D}$ numerically for discrete sets of sources and values of $\omega$. For each source, we cross-correlate the Green’s function calculated at $R1$ with that calculated at $R2$. We stack the resulting cross-correlations in the time-domain, which is equivalent to calculating the $\varphi$- and $\theta$-integrals in the equations above. The stacks can be compared with predictions based on the analytical formulae that we shall illustrate below.
4.1 Ring of sources

The setup associated with eq. (8) is implemented numerically, placing 720 equally-spaced sources along the planar ring centered at $R_1$. $R_1$ and $R_2$ are 20km away from one another and wave speed is 2km/s. We implement eq. (8) directly at frequencies between $\pm$10Hz, with sampling rate 20Hz; we taper the highest and lowest frequencies to avoid ringing artifacts, and inverse-Fourier-transform via numerical FFT [Cooley and Tukey, 1965]. The corresponding cross-correlations are shown in the “gather” plot of Fig. 2a. Sources aligned with the two receivers (azimuth 0° or 180°) result in the longest delay time between arrival of the impulse at $R_1$ and $R_2$: cross-correlation is nonzero near $\pm$10s, which corresponds to the propagation time between $R_1$ and $R_2$. Sources at azimuth around $\sim$90° and $\sim$270° are approximately equidistant from $R_1$ and $R_2$, resulting in the impulse hitting $R_1$ and $R_2$ simultaneously, and cross-correlation in Fig. 2a being nonzero at $\sim$0s for those source azimuths.

The result of averaging over all sources is shown in Fig. 2b. It is clear that the imaginary part of the time-domain signal is 0, as it should be. As for the real part, cross-correlations associated with sources at azimuth 0° or 180° add up constructively; the cumulative contribution of other sources is smaller, but non negligible. The source-averaged cross-correlation is accordingly dominated by two peaks corresponding to energy traveling in a straight path from $R_2$ to $R_1$ (generated by sources at azimuth 0°) and from $R_1$ to $R_2$ (sources at 180°). The two peaks are often labeled “causal” and (not quite appropriately) “anticausal” (or “acausal”), respectively. In this particular case, the causal and anticausal peaks have different amplitude. This asymmetry reflects the asymmetry in the locations of $R_1$ and $R_2$ with respect to the source distribution: $R_2$ is closer to sources at $\varphi = 0^\circ$ than $R_1$ is to sources at $\varphi = 180^\circ$, and waves hitting $R_2$ have accordingly larger amplitude. The description of seismic/acoustic interferometry in terms of a stationary-phase integral (sec. 5.1 and appendix A) explains mathematically why sources at 0° or 180° are most relevant, and provides an analytical relationship between the source-averaged cross-correlation in Fig. 2b and the Green’s function of the medium.

4.2 Sources on a sphere

We next distribute $\sim$$10^3$ approximately equally-spaced sources over the surface of a sphere of radius $R = 100$km, centered at $R_1$. $R_2$ lies 20km away from $R_1$. In analogy with sec. 4.1, the signal recorded at $R_1$ and that recorded at $R_2$ (eq. (201)) are cross-correlated for each source, and cross-correlations are stacked, implementing eq. (9). The source-averaged
Figure 2: Cross-correlations associated with a circular, planar distribution of sources surrounding the two receivers. $R_1$ is at the center of a circle of sources with radius $R=100$km. $R_2$ lies on the plane defined by $R_1$ and the sources, $\Delta=20$km away from $R_1$. Phase velocity $c=2$km/s. Source density $n_C(\varphi) = 2$ sources per degree, independent of $\varphi$. $G_{3D}(r_{1S},\omega)$ and $G_{3D}(r_{2S},\omega)$ are evaluated numerically and cross-correlated for a ring of equally-spaced sources. The result is inverse-Fourier-transformed, and averaged over all sources. (a) Time-domain single-source cross-correlations for all azimuths. (b) Real (solid line) and imaginary (dashed) parts of the source-averaged, time-domain cross-correlation, resulting from stacking the traces in (a). The dimension is that of $G_{3D}(r,\omega)$, squared and integrated over frequency (inverse-Fourier transformed).
Figure 3: Real (solid line) and imaginary (dashed) parts of the average time-domain cross-correlation, resulting from stacking all cross-correlations associated with a uniformly dense distribution of sources over a sphere. \( R_1 \) is at the center of a sphere of sources with radius \( R=100\text{km} \). \( R_2 \) lies \( \Delta=20\text{km} \) away from \( R_1 \). Phase velocity \( c=2\text{km/s} \). The calculation is conducted numerically (sec. 4.2), by evaluating \( G_{3D}(r_{1S},\omega) \) and \( G_{3D}(r_{2S},\omega) \) for each source via eq. (201), and subsequently applying cross-correlation, inverse-Fourier-transformation, and averaging over all sources.

Frequency-domain cross-correlation is computed between \( \pm 10\text{Hz} \), with sampling rate 20Hz, and tapered at high frequency to avoid artifacts. The time-domain result is shown in Fig. 3; in this case, no “gather” plot was made because of the difficulty of visualization when sources are distributed in three dimensions. The source-averaged cross-correlation is real, as it should be; it is different from that of Fig. 2b, in that cross-correlations associated with sources at all azimuths add up constructively, resulting in a boxcar function. Again, the maximum delay time between arrival of the impulse at \( R_1 \) and \( R_2 \) (10s in our setup) corresponds to sources at \( 0^\circ, 180^\circ \), and no energy is recorded at \( t > 10\text{s} \); cross-correlations are accordingly 0 at \( t < -10\text{s} \) and \( t > 10\text{s} \). Despite the asymmetry in receiver location with respect to the sources, the stacked cross-correlation is now symmetric. All these results are reproduced analytically in sec. 5.2.

4.3 Membrane waves/Rayleigh-wave potential

The setup associated with eq. (10) is also reproduced numerically, placing 720 sources at 0.5\(^\circ\) azimuth \( \varphi \) intervals along the ring centered at \( R_1 \). Frequency varies between \( \pm 10\text{Hz} \) with sampling rate=20Hz, high frequencies are tapered as above. As we shall show analytically in sec. 5.3, eq. (10) leads to a non-convergent \( \omega \)-integral when a time-domain expression...
Figure 4: Cross-correlations, differentiated with respect to time, associated with a circular, planar distribution of sources surrounding the two receivers in two dimensions, i.e. on a membrane. The setup is the same as in Fig. 2. $G_{2D}(r_{1S}, \omega)$ and $G_{2D}(r_{2S}, \omega)$ are evaluated numerically and cross-correlated for a ring of 720 equally-spaced sources. The result is inverse-Fourier-transformed, and averaged over all sources. (a) Time-domain single-source cross-correlations for all azimuths. (b) Real (solid line) and imaginary (dashed) parts of the stacked time-derivative of the cross-correlation, resulting from stacking the traces in (a).
for $I_{MW}$ is sought; we therefore compute the time-derivative of $I_{MW}$ (multiplication by $i\omega$ in the frequency domain) before inverse-Fourier transforming. In the interest of speed, we implement the asymptotic approximation of $H_{(2)}^0$ (i.e., the second line of eq. (10)) rather than $H_{(2)}^0$ itself. The results are shown in Fig. 4. The “gather” (Fig. 4a) is qualitatively similar to that of Fig. 2a, but Fig. 4b shows that cross-correlations associated with sources away from azimuth $0^\circ$ and $180^\circ$ cancel out when stacking. After stacking (Fig. 4b) we verify that the time-derivative of $I_{MW}$ is real. We also find that it is antisymmetric with respect to time, indicating that $I_{MW}$ is symmetric, in agreement with the results of sec. 4.2 but not with sec. 4.1.

5 Analytical formulae for 2-receiver cross-correlations of a diffuse wave field

As pointed out by Snieder [2004], integration with respect to $\theta$ and $\varphi$ in eqs. (8)-(10) and (15) can also be conducted analytically by means of the stationary-phase approximation, the details of which are given in appendix A. The numerical results of sec. 4 will serve as a reference to validate the approximate results presented below.

5.1 Ring of sources in free space

The integrand at the right-hand side of eq. (8) coincides with that in (150), after replacing $\lambda=\omega$, $x=\varphi$ and defining $f(\varphi)=n_C(\varphi)/\left[4\pi^2 r_{1S}(\varphi) r_{2S}(\varphi)\right]$ and $\psi(\varphi)=|r_{1S}(\varphi) − r_{2S}(\varphi)|/c$. We shall first identify the values of $\varphi$ such that $\psi'(\varphi)=0$ (stationary points), then use eq. (151) to evaluate the contribution of each stationary point to the integral, and finally combine them. Importantly, the stationary phase approximation is valid at high frequencies $\omega \rightarrow \infty$ (corresponding to $\lambda \rightarrow \infty$ in sec. A.1). Choosing the reference frame as described in sec. 3 (Fig. 1), by definition $r_{1S}$ is constant (we shall call it $R$) and, based on some simple geometrical considerations,

$$r_{2S} = \left[(R \cos \varphi − \Delta)^2 + R^2 \sin^2 \varphi\right]^{\frac{1}{2}} = \left(R^2 + \Delta^2 - 2\Delta R \cos \varphi\right)^{\frac{1}{2}}, \quad (16)$$

where $\Delta$ is interstation distance and $\Delta < R$ in our setup. It follows that

$$f(\varphi) = \frac{1}{(4\pi c)^2} \frac{n_C(\varphi)}{4\pi^2 R \left(R^2 + \Delta^2 - 2\Delta R \cos \varphi\right)^{\frac{1}{2}}} \quad (17)$$
\[
\psi(\varphi) = \frac{1}{c}(r_{1S} - r_{2S}) = \frac{1}{c}\left[(R^2 + \Delta^2 - 2\Delta R \cos \varphi)^{\frac{3}{2}} - R\right], \quad (18)
\]

and upon differentiating \(\psi\) with respect to \(\varphi\),

\[
\psi'(\varphi) = \frac{\Delta R \sin \varphi}{c(R^2 + \Delta^2 - 2\Delta R \cos \varphi)^{\frac{3}{2}},} \quad (19)
\]

\[
\psi''(\varphi) = \frac{\Delta R \cos \varphi}{c(R^2 + \Delta^2 - 2\Delta R \cos \varphi)^{\frac{3}{2}}} - \frac{\Delta^2 R^2 \sin^2 \varphi}{c(R^2 + \Delta^2 - 2\Delta R \cos \varphi)^{\frac{3}{2}}}. \quad (20)
\]

We infer from (19) that the stationary points of \(\psi(\varphi)\) within the domain of integration are \(\varphi = -\pi, 0, \pi\) (corresponding to \(\sin \varphi = 0\)). Following Bender and Orszag [1978], we rewrite eq. (8) as a sum of integrals limited to the vicinity of stationary points, and with a stationary point as one of the integration limits:

\[
\int_C(\omega) \approx \int_{-\pi}^{\pi} f(\varphi) e^{i\omega \psi(\varphi)} d\varphi + \int_{0}^{\pi} f(\varphi) e^{i\omega \psi(\varphi)} d\varphi + \int_{-\pi}^{0} f(\varphi) e^{i\omega \psi(\varphi)} d\varphi + \int_{\pi}^{0} f(\varphi) e^{i\omega \psi(\varphi)} d\varphi, \quad (21)
\]

which is valid for \(\omega \rightarrow \infty\) and arbitrarily small \(\varepsilon\). Eq. (151) can now be used directly to integrate each of the terms at the right-hand side of (21), and after noticing that both \(f(\varphi)\) and \(\psi(\varphi)\) are symmetric with respect to \(\varphi = 0\) we find

\[
\int_C(\omega) \approx 2f(\pi)e^{i(\omega \psi(\pi)\pm \frac{\pi}{2})} \sqrt{\frac{\pi}{2\omega|\psi''(\pi)|}} + 2f(0)e^{i(\omega \psi(0)\pm \frac{\pi}{2})} \sqrt{\frac{\pi}{2\omega|\psi''(0)|}}. \quad (22)
\]

We now need to evaluate \(f\), \(\psi\) and \(\psi''\) at 0 and \(\pi\). It follows from the definition of \(f\), from eqs. (18) and (20), and from the fact that \(R > \Delta\) (and hence \(|R - \Delta| = R - \Delta\)) that

\[
f(0) = \frac{1}{(4\pi c)^2} \frac{n_C(0)}{4\pi^2 R(R - \Delta)}, \quad (23)
\]

\[
f(\pm \pi) = \frac{1}{(4\pi c)^2} \frac{n_C(\pm \pi)}{4\pi^2 R(R + \Delta)}, \quad (24)
\]

\[
\psi(0) = -\frac{\Delta}{c}, \quad (25)
\]

\[
\psi(\pm \pi) = \frac{\Delta}{c}, \quad (26)
\]

\[
\psi''(0) = \frac{\Delta R}{c(R - \Delta)}, \quad (27)
\]

\[
\psi''(\pm \pi) = -\frac{\Delta R}{c(R + \Delta)}. \quad (28)
\]

To obtain eqs. (25) through (28), one must also recall that \(\sqrt{R^2 + \Delta^2 - \Delta R \cos \varphi} = r_{2S}\) is,
physically, a positive distance: when \( \varphi = 0, \pm \pi \), it follows that \( \sqrt{R^2 + \Delta^2 - 2\Delta R} = R - \Delta \).

Substituting expressions (23) through (28) into eq. (22), we find after some algebra that

\[
I_C(\omega) \approx \frac{1}{(4\pi \varepsilon)^2} \frac{1}{2\pi} \sqrt{\frac{c}{2\pi \Delta R^3}} \left[ n_C(\pi) \frac{e^{i(\varphi\Delta - \frac{\pi}{4})}}{\sqrt{(R + \Delta)\omega}} + n_C(0) \frac{e^{-i(\varphi\Delta - \frac{\pi}{4})}}{\sqrt{(R - \Delta)\omega}} \right],
\]

where the sign of \( \pi/4 \) in the argument of the exponential function was selected based on the sign of \( \psi'' \) as explained in sec. A.1. Comparing eq. (29) with (196), it is apparent that both terms at the right-hand side of (29) are proportional to the high-frequency/far-field form of the Green’s function \( G_{2D}(\Delta, \omega) \) (and, interestingly, not \( G_{3D} \)).

Equation (21) and the subsequent expressions for \( I_C(\omega) \) are only valid for large and positive \( \omega \), so that \( I_C(\omega) \) remains undefined for \( \omega < 0 \). We know, however, that a sum of cross-correlations of real-valued functions of time should be real-valued in the time domain, requiring that \( I_C(\omega) = I_C^*(-\omega) \), and

\[
\mathcal{F}^{-1}[I_C(\omega)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega I_C(\omega) e^{i\omega t} = \frac{1}{\sqrt{2\pi}} \left( \int_{0}^{+\infty} d\omega I_C(\omega) e^{i\omega t} + \int_{0}^{-\infty} d\omega I_C^*(\omega) e^{-i\omega t} \right)
\]

After substituting (29) into (30), the integration over \( \omega \) can be conducted analytically, making
use of the identity
\[
\int_0^\infty dx \frac{e^{iax}}{\sqrt{x}} = \begin{cases} 
\frac{1}{\sqrt{ai}} \sqrt{\frac{\pi}{2}} (1 + i) & \text{if } a > 0, \\
\frac{1}{\sqrt{-ai}} \sqrt{\frac{\pi}{2}} (1 - i) & \text{if } a < 0,
\end{cases}
\] (31)

and integrating separately for the three cases \( t < -\frac{\Delta}{c} \), \(-\frac{\Delta}{c} < t < \frac{\Delta}{c} \) and \( t > \frac{\Delta}{c} \). After a considerable amount of algebra we find
\[
I_C(t) = \frac{1}{(4\pi c)^2} \frac{1}{2\pi} \sqrt{\frac{c}{\pi \Delta R^3}} \times \begin{cases} 
nC(0) \frac{1}{\sqrt{R - \Delta}} \sqrt{\frac{\Delta}{\pi}} - t & \text{if } t < -\frac{\Delta}{c}, \\
nC(\pi) \frac{1}{\sqrt{R + \Delta}} \sqrt{\frac{\Delta}{\pi}} + t & \text{if } -\frac{\Delta}{c} < t < \frac{\Delta}{c}, \\
nC(\pi) \frac{1}{\sqrt{R + \Delta}} \sqrt{\frac{\Delta}{\pi}} + t & \text{if } t > \frac{\Delta}{c},
\end{cases}
\] (32)

which, as required, has zero imaginary part. We infer from eq. (32) that, in the current setup, there is no explicit relationship between the source-averaged cross-correlation \( I_C(t) \) and the Green’s functions \( G_{2D} \) or \( G_{3D} \) (App. E). Fig. 5 shows that \( I_C(t) \) as obtained from eq. (32) is consistent with the numerical result of Fig. 2. The discussion of sec. 4.1 remains valid. A slight discrepancy between the “analytical” and “numerical” results in Fig. 5 is explained by the stationary-phase approximation being strictly valid only at high frequency \( (\omega \to \infty) \).

5.2 Sources over a spherical surface

The integral in eq. (9) can be solved analytically with the help of eq. (161). Equation (9) is indeed a particular case of (152), with \( x = \theta \), \( y = \varphi \), \( \lambda = \omega \), \( f(\theta, \varphi) = n_S \sin \theta /[8\pi^2 (r_{2S}(\theta, \varphi) - r_{1S}(\theta, \varphi))] \) and \( \psi(\theta, \varphi) = [r_{2S}(\theta, \varphi) - r_{1S}(\theta, \varphi)]/c \). Again, the reference frame is centered on \( R1 \), so that \( r_{1S}(\theta, \varphi) = R \) for all values of \( \theta, \varphi \). After some algebra, we find
\[
r_{2S} = \left( R^2 + \Delta^2 - 2\Delta R \sin \theta \cos \varphi \right)^{\frac{1}{2}},
\] (33)

and consequently
\[
f(\theta, \varphi) = \frac{1}{(4\pi c)^2} \frac{n_S(\theta, \varphi) \sin \theta}{8\pi^2 R \left( R^2 + \Delta^2 - 2\Delta R \sin \theta \cos \varphi \right)^{\frac{1}{2}}},
\] (34)

\[
\psi(\varphi) = \frac{1}{c} (r_{2S} - r_{1S}) = \frac{1}{c} \left[ \left( R^2 + \Delta^2 - 2\Delta R \sin \theta \cos \varphi \right)^{\frac{1}{2}} - R \right].
\] (35)
Differentiating (35) with respect to $\theta$ and $\varphi$ we find

$$
\psi_\theta = -\frac{\Delta R \cos \theta \cos \varphi}{c \left( R^2 + \Delta^2 - 2\Delta R \sin \theta \cos \varphi \right)^{3/2}},
$$

(36)

where the compact $\psi_\theta$ stands for $\frac{\partial \psi}{\partial \theta}$. Likewise,

$$
\psi_\varphi = \frac{\Delta R \sin \theta \sin \varphi}{c \left( R^2 + \Delta^2 - 2\Delta R \sin \theta \cos \varphi \right)^{3/2}}.
$$

(37)

If we continue differentiating,

$$
\psi_{\theta\theta} = -\frac{\Delta R \cos^2 \theta \cos^2 \varphi}{c \left( R^2 + \Delta^2 - 2\Delta R \sin \theta \cos \varphi \right)^{3/2}} + \frac{\Delta R \sin \theta \cos \varphi}{c \left( R^2 + \Delta^2 - 2\Delta R \sin \theta \cos \varphi \right)^{1/2}},
$$

(38)

$$
\psi_{\varphi\varphi} = -\frac{\Delta R \sin^2 \theta \sin^2 \varphi}{c \left( R^2 + \Delta^2 - 2\Delta R \sin \theta \cos \varphi \right)^{3/2}} + \frac{\Delta R \sin \theta \cos \varphi}{c \left( R^2 + \Delta^2 - 2\Delta R \sin \theta \cos \varphi \right)^{1/2}},
$$

(39)

$$
\psi_{\theta\varphi} = \frac{\Delta R \sin \theta \sin \varphi \cos \theta \cos \varphi}{c \left( R^2 + \Delta^2 - 2\Delta R \sin \theta \cos \varphi \right)^{3/2}} + \frac{\Delta R \cos \theta \sin \varphi}{c \left( R^2 + \Delta^2 - 2\Delta R \sin \theta \cos \varphi \right)^{1/2}},
$$

(40)

Equations (36) and (37) allow us to identify the stationary points $(\theta, \varphi)$ such that $\psi_\theta, \psi_\varphi = 0, 0$ of the integrand at the right hand side of (9): namely $(\theta=0, \varphi=\pm \pi/2), (\pi, \pm \pi/2), (\pi/2, 0), (\pi/2, \pi)$. It is sufficient to evaluate (161) at these points, and sum the results, to find an analytical expression for the integral (9). We notice first of all that $f(\theta, \varphi)=0$ if $\theta = 0, \pm \pi$: the corresponding stationary points give no contribution to the integral and will be neglected in the following. We are left with the stationary points at $\theta = \pi/2$ and $\varphi=0, \pi$.

Let us evaluate $f$, $\psi$, $\psi_{\theta\theta}$, $\psi_{\varphi\varphi}$, $\psi_{\theta\varphi}$ at those points.

$$
f \left( \frac{\pi}{2}, 0 \right) = \frac{1}{(4\pi c)^2} \frac{n_S \left( \frac{\pi}{2}, 0 \right)}{8\pi^2 R (R - \Delta)},
$$

(41)

$$
f \left( \frac{\pi}{2}, \pi \right) = \frac{1}{(4\pi c)^2} \frac{n_S \left( \frac{\pi}{2}, \pi \right)}{8\pi^2 R (R + \Delta)},
$$

(42)

$$
\psi \left( \frac{\pi}{2}, 0 \right) = -\frac{\Delta}{c},
$$

(43)

$$
\psi \left( \frac{\pi}{2}, \pi \right) = \frac{\Delta}{c},
$$

(44)

$$
\psi_{\theta\theta} \left( \frac{\pi}{2}, 0 \right) = \frac{\Delta R}{c(R - \Delta)},
$$

(45)

$$
\psi_{\theta\theta} \left( \frac{\pi}{2}, \pi \right) = -\frac{\Delta R}{c(R + \Delta)},
$$

(46)
\[ \psi_{\varphi\varphi} \left( \frac{\pi}{2}, 0 \right) = \frac{\Delta R}{c(R - \Delta)}, \quad (47) \]
\[ \psi_{\varphi\varphi} \left( \frac{\pi}{2}, \pi \right) = -\frac{\Delta R}{c(R + \Delta)}, \quad (48) \]
and it follows immediately from (40) that \( \psi_{\theta\varphi} = 0 \) at all stationary points.

The above expressions can be substituted into (161) to find

\[ I_S(\omega) \approx \frac{1}{(4\pi c)^2} \frac{ic}{4\pi R^2 \Delta} \left[ n_S \left( \frac{\pi}{2}, 0 \right) e^{-i\frac{\omega \Delta}{c}} - n_S \left( \frac{\pi}{2}, \pi \right) e^{i\frac{\omega \Delta}{c}} \right], \quad (49) \]
which satisfies \( I_S(-\omega) = -I_S^*(\omega) \). Notice that, substituting eq. (201) into (49),

\[ I_S(\omega) \approx \frac{\sqrt{2\pi}}{(4\pi R)^2} \frac{i}{c} \left[ n_S \left( \frac{\pi}{2}, 0 \right) G_{3D}(\Delta, \omega) - n_S \left( \frac{\pi}{2}, \pi \right) G_{3D}^*(\Delta, \omega) \right], \quad (50) \]
and if \( n_S \left( \frac{\pi}{2}, 0 \right) = n_S \left( \frac{\pi}{2}, \pi \right) \), eq. (50) can be further simplified to

\[ I_S(\omega) \approx -\frac{\sqrt{2\pi}}{8\pi^2 R^2} n_S \left( \frac{\pi}{2}, 0 \right) \frac{1}{\omega} \Im \left[ G_{3D}(\Delta, \omega) \right]. \quad (51) \]

We next inverse-Fourier transform eq. (49) to the time domain. In our convention,

\[ \mathcal{F}^{-1} \left( \frac{1}{\omega} \right) = i \sqrt{\frac{\pi}{2}} \text{sgn}(t) \quad (52) \]
(Dirichlet integral), where the sign function \( \text{sgn} \) is +1 or -1 for positive and negative values of the argument, respectively. In the time domain, \( I_S \) is then

\[ I_S(t) \approx \frac{1}{(4\pi c)^2} \frac{c}{4\sqrt{2\pi} R^2 \Delta} \left[ n_S \left( \frac{\pi}{2}, \pi \right) \text{sgn} \left( t + \frac{\Delta}{c} \right) - n_S \left( \frac{\pi}{2}, 0 \right) \text{sgn} \left( t - \frac{\Delta}{c} \right) \right], \quad (53) \]
illustrated in Fig. 6. This result is consistent with eq. (15) of Roux et al. [2005] and eq. (27) of Nakahara [2006], who treated the same physical problem in different ways, and with Fig. 1a of Harmon et al. [2008]. By comparison with eq. (200), it is apparent that the terms at the right-hand side of eq. (53) are time-integrals of the time-domain Green’s function \( G_{3D}(\Delta, t) \).

Equations (49) and (53) mean that the Green’s function \( G_{3D} \) corresponding to propagation from receiver \( R_1 \) to \( R_2 \) and vice-versa can be found by (i) recording at both receivers the signal emitted by a dense distribution of sources covering all azimuths in three dimensions; (ii) cross-correlating, for each source, the signal recorded at \( R_1 \) with that recorded at \( R_2 \); (iii) integrating the cross-correlation over the source location \( r \).
Figure 6: Stacked cross-correlation resulting from a spherical distribution of sources surrounding the two receivers, predicted (green line) by the approximate analytical formula (53), and (black) computed numerically as described in sec. 4.2 and already illustrated in Fig. 3 (only the real part is shown here). The source/station setup is the same as for Fig. 3.

Our eq. (49) coincides with eq. (11) of Snieder [2004], except that here the difference between the right-hand-side terms is taken, while in Snieder [2004] they are summed. Inverse-Fourier transforming the formula of Snieder [2004] would lead to a non-physical cross-correlation, nonzero at times where cross-correlation is necessarily zero at all $\varphi$. We infer that eqs. (11) and (12) in Snieder [2004] are wrong. However, this error did not affect the subsequent, Rayleigh-wave derivation of Snieder [2004].

5.3 Membrane (or Rayleigh) waves from 2-D distribution of point sources

5.3.1 Source-averaged cross-correlation

We treat the integral in eq. (10) with the stationary-phase method as illustrated in previous sections. The phase term $\psi(\varphi)$ in (10) is the additive inverse of that in (8), and the two integrands share the same stationary points $\varphi=0, \varphi=\pm \pi$. At those points,

$$f(0) = \frac{n_M(0)}{32\pi^3 c^3 \omega \sqrt{R(R-\Delta)}}, \quad \text{(54)}$$

$$f(\pm \pi) = \frac{n_M(\pi)}{32\pi^3 c^3 \omega \sqrt{R(R+\Delta)}}, \quad \text{(55)}$$

$$\psi(0) = \frac{\Delta}{c}, \quad \text{(56)}$$

$$\psi(\pm \pi) = -\frac{\Delta}{c}, \quad \text{(57)}$$
\[ \psi''(0) = -\frac{\Delta R}{c(R - \Delta)}, \quad (58) \]
\[ \psi''(\pm \pi) = \frac{\Delta R}{c(R + \Delta)}, \quad (59) \]

Substituting into eq. (151) (which is multiplied by 2 for each stationary point) and summing over our set of two stationary points,

\[ I_{MW}(\omega) \approx \frac{1}{16\pi^3 c^2 R} \sqrt{\frac{\pi c}{2\Delta}} \omega^{-1/2} \left[ n_M(0)e^{i\left(\frac{\pi \Delta}{c} - \frac{\omega}{2}\right)} + n_M(\pi)e^{-i\left(\frac{\pi \Delta}{c} - \frac{\omega}{2}\right)} \right], \quad (60) \]

coheraently with eqs. (23) and (24) of Snieder [2004], which however involve summation over normal modes. Comparing eq. (60) with (196), we can also write

\[ I_{MW}(\omega) \approx \frac{i}{4\pi^2 \omega c R} \sqrt{\frac{\pi}{2}} \left[ n_M(\pi)G_{2D}(\Delta, \omega) - n_M(0)G_{2D}^*(\Delta, \omega) \right], \quad (61) \]

valid in the asymptotic (high-frequency/far-field) approximation.

Eqs. (60) and (61) are valid in the stationary-phase approximation, i.e. for large and positive \( \omega \) only. We know, however, that \( I_{MW}(t) \) must be real: as in sec. 5.1, \( I_{MW} \) at \( \omega < 0 \) is thus defined by \( I_{MW}(\omega) = I_{MW}(-\omega) \), and eq. (30) remains valid after replacing \( I_C \) with \( I_{MW} \), i.e.

\[ \mathcal{F}^{-1}[I_{MW}(\omega)] = \frac{1}{\sqrt{2\pi}} \left( \int_0^{+\infty} d\omega \ I_{MW}(\omega)e^{i\omega t} + \int_0^{+\infty} d\omega \ I_{MW}^*(\omega)e^{-i\omega t} \right). \quad (62) \]

After substituting the expression (60) for \( I_{MW} \) into (62), it becomes apparent that finding a time-domain expression for \( I_{MW} \) requires the solution of a non-convergent integral, namely

\[ \int_0^{+\infty} \int_0^{+}\int_0^{+\infty} dx \ x^{-3/2} \cos(x). \]

5.3.2 Derivative of the source-averaged cross-correlation

One can still use the present theoretical formulation to interpret 2-D data in the time domain, by simply taking the time-derivative of both the observed source-averaged cross-correlation and its analytical expression \( I_{MW}(\omega) \). Based on the properties of Fourier transforms, the latter is achieved by multiplying \( I_{MW}(\omega) \) by \( i\omega \),

\[ I'_{MW}(\omega) \approx \frac{i}{16\pi^3 c^3 R} \sqrt{\frac{\pi c}{2\Delta}} \left[ \frac{n_M(0)e^{i\left(\frac{\pi \Delta}{c} - \frac{\omega}{2}\right)}}{\sqrt{\omega}} + \frac{n_M(\pi)e^{-i\left(\frac{\pi \Delta}{c} - \frac{\omega}{2}\right)}}{\sqrt{\omega}} \right]. \quad (63) \]

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We infer from eq. (63) that for acoustic waves in 2-D, and within the stationary-phase approximation, the derivative of the source-averaged cross-correlation is proportional to the sum of causal and anticausal Green’s functions \( G_{2D}(\omega) \) given by eq. (196) [Snieder, 2004].

Like (60), eq. (63) is only valid for large positive \( \omega \), but the fact that \( I'_{MW}(t) \) is real can be used to define \( I'_{MW}(\omega) \) at \( \omega < 0 \) via eq. (30). The inverse Fourier transform of the resulting expression for \( I'_{MW}(\omega) \) involves a convergent integral (eq. (31)) and can be found analytically. The procedure is similar to that of sec. 5.1, eqs. (30) through (32). The result is quite different, as can be expected since the imaginary unit multiplies the right-hand side of eq. (63). Namely,

\[
I'_{MW}(t) \approx \frac{1}{16 \pi c^3 R} \sqrt{\frac{\pi c}{\Delta}} \times \begin{cases} 
\frac{n_M(0)}{\sqrt{t-\frac{\Delta}{c}}} & \text{if } t < -\frac{\Delta}{c}, \\
0 & \text{if } -\frac{\Delta}{c} < t < \frac{\Delta}{c}, \\
-\frac{n_M(\omega)}{\sqrt{t-\frac{\Delta}{c}}} & \text{if } t > \frac{\Delta}{c}.
\end{cases}
\]  

(64)

Fig. 7 shows that eq. (64) is validated by the numerical results of sec. 4.3. No explicit mathematical relationship between \( I'_{MW} \) and \( G_{2D} \) can be inferred from eq. (64), although Fig. 7 shows that the behaviour of the two functions is qualitatively similar.
5.3.3 Symmetric source distribution

Inverse Fourier transformation turns out to be simpler when \( n_M(\pi) = n_M(0) \): setting them both to 1 for simplicity, eq. (61) simplifies to

\[
I_{MW}(\omega) \approx \frac{i}{4\pi^2 \omega c R} \sqrt{\frac{\pi}{2}} \left[ G_{2D}(\Delta, \omega) - G_{2D}^*(\Delta, \omega) \right]
\]

and

\[
I_{MW}(t) \approx -\frac{1}{2\pi^2 \omega c R} \sqrt{\frac{\pi}{2}} \Im \left[ G_{2D}(\Delta, \omega) \right].
\]

Let us denote \( G_{2D}^o \) the odd function \( G_{2D}^o(t) = \frac{1}{2} G_{2D}(t) - \frac{1}{2} G_{2D}(-t) \). We know from appendix B, eq. (170), that \( \Im [G_{2D}(\Delta, \omega)] = -iG_{2D}^o(\Delta, \omega) \), so that

\[
I_{MW}(t) \approx -\frac{1}{2\pi^2 c R} \sqrt{\frac{\pi}{2}} \int_{\tau = -\infty}^{\tau = \tau} \left[ G_{2D}^o(\tau) d\tau \right]
\]

and by definition of \( G_{2D}^o \),

\[
I_{MW}(t) \approx -\frac{1}{2\pi^2 c R} \sqrt{\frac{\pi}{2}} \int_{\tau = -\infty}^{\tau = \tau} \left[ G_{2D}(\tau) - G_{2D}(-\tau) \right] d\tau.
\]

Differentiating with respect to time,

\[
I'_{MW}(t) \approx \frac{1}{4\pi^2 c R} \sqrt{\frac{\pi}{2}} \left[ G_{2D}(-t) - G_{2D}(t) \right],
\]

consistent with Fig. 7. The apparent discrepancy between eqs. (64) and (70) is explained by the fact that the derivation of (70) involved identifying \( G_{2D} \) with its asymptotic approximation (196), which was not the case for the derivation of eq. (64).
5.4 Plane waves

5.4.1 Exact integration

An elegant way of reducing eq. (15) to a simple and useful identity is to compare it with the integral form of the 0th-order Bessel function of the first kind,

\[ J_0(z) = \frac{1}{\pi} \int_0^\pi d\varphi \cos(z \cos \varphi) \]  \hspace{1cm} (71)

[Abramowitz and Stegun, 1964, eq. (9.1.18)]. Substituting (71) into (15),

\[ I_{PW}(\omega) = \sqrt{\frac{\pi}{2}} q(\omega) J_0 \left( \frac{\omega \Delta}{c} \right), \]  \hspace{1cm} (72)

an exact equality that does not require the stationary-phase approximation. Importantly, eq. (72) was originally obtained in the early study of Aki [1957], providing the basis for much of the later work in ambient-noise seismology. Comparing (72) with (195), we infer that the source-averaged cross correlation \( I_{PW}(\omega) \) is proportional to the imaginary part of the membrane-wave Green’s function \( G_{2D} \left( \frac{r}{c} \right) \) if \( r = \Delta \).

5.4.2 Approximate integration

In analogy with previous sections, the integral in eq. (15) can also be solved by means of the stationary phase approximation. Let us rewrite it

\[ I_{PW}(\omega) = \frac{q(\omega)}{\sqrt{2\pi}} \Re \left[ \int_0^\pi d\varphi e^{i(\varphi \Delta \cos \varphi)} \right]. \]  \hspace{1cm} (73)

The integral at the right-hand side of eq. (73) coincides with that in (150) after replacing \( a = 0, b = \pi, x = \varphi, \lambda = \omega \) and \( \psi(\varphi) = \frac{\Delta \cos \varphi}{c} \). Taking the \( \varphi \)-derivative of \( \psi \),

\[ \psi_\varphi(\varphi) = -\frac{\Delta}{c} \sin \varphi, \]  \hspace{1cm} (74)

and we see immediately that there are two stationary points within the integration domain, at the integration limits \( \varphi=0,\pi \). Differentiating again, we find

\[ \psi_{\varphi\varphi}(\varphi) = -\frac{\Delta}{c} \cos \varphi. \]  \hspace{1cm} (75)
At the stationary point $\varphi = 0$ we have $\psi(0) = \Delta c$ and $\psi_{\varphi\varphi}(0) = -\Delta c$. At $\varphi = \pi$, $\psi(0) = -\Delta c$ and $\psi_{\varphi\varphi}(0) = \Delta c$. For each stationary point, we substitute the corresponding values into (151), choosing, as usual, the sign of $\pi/4$ in the argument of the exponential based on that of $\psi_{\varphi\varphi}$. We next sum the contributions of both stationary points, finding

$$
\int_0^{\pi} d\varphi e^{i \omega \Delta \cos \varphi} \approx \sqrt{\frac{2\pi}{\omega \Delta}} \cos \left( \frac{\omega \Delta c - \pi}{4} \right). \tag{76}
$$

Substituting, in turn, (76) into (73),

$$
I_{PW}(\omega) \approx q(\omega) \sqrt{\frac{c}{\omega \Delta}} \cos \left( \frac{\omega \Delta c - \pi}{4} \right) \approx \sqrt{\frac{\pi}{2}} q(\omega) J_0 \left( \frac{\omega \Delta c}{c} \right), \tag{77}
$$

where we have used the asymptotic approximation of $J_0$ [Abramowitz and Stegun, 1964, eq. (9.2.1)], valid in the high-frequency (and/or far-field) limit, i.e. in the range of validity of the stationary-phase approximation. Eq. (77) shows that the stationary-phase approximation leads to an estimate of source-averaged cross-correlation $I_{PW}(\omega)$ consistent, at large $\omega$, with the result (72).

The relationship (72) between observed stacked cross-correlations and the Bessel’s function $J_0$ has been applied e.g. by Ekström et al. [2009] and Ekström [2014] to analyze ambient-noise surface-wave data in the frequency domain, and measure their velocity. The inverse-Fourier transform of (72) is obtained and discussed by Nakahara [2006], who also shows that the Hilbert transform of the stacked cross-correlation coincides with the (causal minus anticausal) $G_{2D}$ [Nakahara, 2006, eq. (19)].

### 5.4.3 Monochromatic plane waves in the time domain

The treatments of Sanchez-Sesma and Campillo [2006], Tsai [2009] and Boschi et al. [2013] are slightly different from the plane-wave formulation presented here, in that they work with monochromatic plane waves in the time domain. Following Tsai [2009], Boschi et al. [2013] make use of the properties of the Bessel and Struve functions to solve the integral in eq. (13), and are eventually able to write the source-averaged cross-correlation of plane waves of frequency $\omega_0$ traveling along all azimuths as

$$
I_{TD}(\omega_0, t) = q(\omega_0) \sqrt{\frac{c}{8\pi\omega_0\Delta}} \left\{ \cos \left[ \omega_0 \left( t + \frac{\Delta}{c} \right) - \frac{\pi}{4} \right] + \cos \left[ \omega_0 \left( t - \frac{\Delta}{c} \right) + \frac{\pi}{4} \right] \right\}. \tag{78}
$$
Boschi et al., 2013, eqs. (35) and (41), valid in the far-field (large $\Delta$) and/or high-frequency (large $\omega_0$) approximations.

To verify that our eq. (77) is consistent with (78), let us take the Fourier transform of the latter. From the equality $\mathcal{F}\{\cos[\omega_0(t + k)]\} = \sqrt{\frac{\pi}{2}} [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] e^{-i\omega k}$ it follows that

$$I_{TD}(\omega_0, \omega) = \frac{q(\omega_0)}{2} \sqrt{\frac{c}{\omega_0 \Delta}} [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] \cos \left( \omega \left( \frac{\pi}{4\omega_0} - \frac{\Delta}{c} \right) \right). \quad (79)$$

Like (14), eq. (79) is valid for any $\omega_0$. Implicitly repeating the monochromatic-wave, time-domain analysis at all $\omega_0$'s,

$$I_{TD}(\omega) = \frac{q(\omega)}{\sqrt{\frac{c}{\omega \Delta}}} \cos \left( \frac{\omega \Delta}{c} - \frac{\pi}{4} \right), \quad (80)$$

which coincides, as expected, with our expression (77) for $I_{PW}(\omega)$.

6 Other derivations

6.1 Time-domain approach

We loosely follow the treatment of Roux et al. [2005] (sec. II), with acoustic sources uniformly distributed throughout an unbounded, infinite 3-D medium. This setup is similar to sec. 5.2 here. Although in sec. 5.2 sources are all lying on the surface of a sphere (azimuthally uniform distribution), we show in the following that the formula we find for the source-averaged cross-correlation is proportional to that derived by Roux et al. [2005].

In free 3-D space, the time-domain cross-correlation between impulsive signals emitted at $S$ and recorded at $R_1$ and $R_2$ (i.e. the time-domain version of eq. (6)) reads

$$\frac{1}{T} \int_{-T/2}^{+T/2} d\tau G_{3D}(r_{1S}, \tau) G_{3D}(r_{2S}, t + \tau) = \frac{1}{(4\pi c)^2 T} \int_{-T/2}^{+T/2} d\tau \frac{\delta \left( \tau - \frac{r_{1S}}{c} \right) \delta \left( \tau + t - \frac{r_{2S}}{c} \right)}{r_{1S} r_{2S}} \quad (81)$$

where we have used expression (200) for $G_{3D}$, and we have limited cross-correlation to a finite time interval $(-T/2, T/2)$, whose length $T$ is related, in practice, to interstation distance and wave speed.

As discussed in sec. 3, if “cross terms” are neglected (app. D), the cross-correlation of a diffuse wave field recorded at $R_1$ and $R_2$ is estimated by summing expression (81) over many uniformly distributed sources $S$. For sources densely distributed throughout the entire 3-D
space $\mathbb{R}^3$,

$$I_{3D}(t) = \frac{1}{(4\pi c)^2} \frac{1}{T} \int_{\mathbb{R}^3} d^3r \frac{\delta \left( t + \frac{r_{1S}}{c} - \frac{r_{2S}}{c} \right)}{r_{1S} r_{2S}},$$  \hspace{1cm} (82)

where $\mathbf{r}$ is the source location, and, said $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, the locations of $R1$ and $R2$, $r_{1S} = |\mathbf{r} - \mathbf{r}_1|$, $r_{2S} = |\mathbf{r} - \mathbf{r}_2|$. Eq. (82) is equivalent to eq. (9) of Roux et al. [2005].

As noted by Roux et al. [2005], eq. (82) shows that a wiggle in the cross-correlation $I_{3D}$ at time $t$ is necessarily associated to one or more sources whose location $\mathbf{r}$ satisfies $ct = |\mathbf{r} - \mathbf{r}_2| - |\mathbf{r} - \mathbf{r}_1|$. In two dimensions, the latter is the equation of a hyperbola with foci at $R1$ at $R2$; in three dimensions, it is the equation of the single-sheet hyperboloid obtained by rotation of said hyperbola around the vertical axis. It follows that the integral in (82) can be reduced to a surface integral over the hyperboloid. After this simplification, Roux et al. [2005] are able to solve the integral in (82) analytically: they find $I_{3D}(t)$ to be a boxcar function between $t = \pm \Delta/c$, equivalent to eq. (49) in sec. 5.2 here.

6.2 Reciprocity-theorem approach

All descriptions of seismic and acoustic interferometry that we discussed so far rest on eq. (1) and on the Green’s function formulae of appendix E, which are strictly valid only for g, infinite media. More general formulations have been developed by Wapenaar [2004], Weaver and Lobkis [2004], van Manen et al. [2005], Wapenaar and Fokkema [2006], Snieder [2007] and others, based on the reciprocity, or Betti’s theorem [e.g., Aki and Richards, 2002, sec. 2.3.2].

6.2.1 acoustic waves

The propagation of acoustic waves in a nonhomogeneous, lossless stagnant gas of density $\rho$ and compressibility $\kappa$ is described by eq. (6.2.7) of Morse and Ingard [1986], plus the forcing term $q$ corresponding to mass injection. In the frequency domain,

$$\nabla p + i\omega \rho \mathbf{v} = 0, \hspace{1cm} (83)$$

$$\nabla \cdot \mathbf{v} + i\omega \kappa p - q = 0. \hspace{1cm} (84)$$

In the more general case of a sound-absorbing medium, $\kappa$ is complex and its imaginary part is proportional to the rate of energy loss (attenuation) [Kinsler et al., 1999, chapter 8]. Eqs. (83) and (84) are equivalent to eqs. (2) and (3) of Wapenaar and Fokkema [2006], or eqs. (1)
and (2) of Snieder [2007] where a different Fourier-transform convention applies. Eq. (83) implies \( v = \frac{1}{i\omega\rho} \nabla p \), and this expression for \( v \) can be substituted into (84), to obtain

\[
\nabla \cdot \left( \frac{1}{\rho} \nabla p \right) + \omega^2 \kappa p = -i\omega q.
\]

(85)

For homogeneous or smooth media, where the spatial derivatives of \( \rho \) and \( \kappa \) are zero or approximately zero, (85) further simplifies to the Fourier transform of eq. (1),

\[
\frac{1}{\rho} \nabla^2 p + \omega^2 \kappa p = -i\omega q.
\]

(86)

Following Wapenaar and Fokkema [2006], let us call \( \mathcal{G} \) the solution of (86) when \( q(x,t) = \delta(x)\delta(t) \):

\[
\frac{1}{\rho} \nabla^2 \mathcal{G} + \omega^2 \kappa \mathcal{G} = -i\omega \delta(x).
\]

(87)

\( \mathcal{G} \) can be interpreted as the Green’s function associated with eq. (84) plus the condition (83). The relationship between \( \mathcal{G} \) and \( G_{3D} \) as defined in app. E can be determined if one considers that eq. (87) is the usual scalar wave equation (whose Green’s function is \( G_{3D} \)) with forcing term \( -i\omega \delta(x) \); based on eq. (213), the time-domain solution to (87) is then the convolution of \( G_{3D} \) with the inverse-Fourier transform of the forcing term \( -i\omega \delta(x) \); in the frequency domain the convolution reduces to the product of the functions in question, and

\[
\mathcal{G} = -i\omega G_{3D}.
\]

(88)

Wapenaar and Fokkema [2006] also introduce a “modified Green’s function” (see their eq. (32)) which coincides with our \( G_{3D} \) except for the sign.

Let us next consider a volume \( V \) bounded by a surface \( \partial V \). (\( \partial V \) is just an arbitrary closed surface within a medium, and generally does not represent a physical boundary.) Let \( q_A(r,\omega), p_A(r,\omega) \) and \( v_A(r,\omega) \) denote a possible combination of forcing, pressure and velocity, respectively, co-existing at \( r \) in \( V \) and \( \partial V \). A different forcing \( q_B \) would give rise, through eqs. (83) and (84), to a different “state” \( B \), defined by \( p_B(r,\omega) \) and \( v_B(r,\omega) \).

A useful relationship between the states \( A \) and \( B \), known as “reciprocity theorem”, is obtained by combining eqs. (83) and (84) as follows,

\[
\int_V d^3r \left[ (83)_A \cdot v_B^* + (83)_B^* \cdot v_A + (84)_A p_B^* + (84)_B^* p_A \right] = 0
\]

(89)
integrals at the left hand side of eq. (94), which then simplifies to

\[ (83)_A \cdot \mathbf{v}_B^* = \nabla p_A \cdot \mathbf{v}_B^* + i \omega \rho \mathbf{v}_A \cdot \mathbf{v}_B^* \]  

(90)

\[ (83)_B^* \cdot \mathbf{v}_A = \nabla p_B^* \cdot \mathbf{v}_A - i \omega \rho \mathbf{v}_B^* \cdot \mathbf{v}_A \]  

(91)

\[ (84)_A p_B^* = \nabla \cdot \mathbf{v}_A p_B^* + i \omega \kappa p_B^* - q_A^* p_B^* \]  

(92)

\[ (84)_B^* p_A = \nabla \cdot \mathbf{v}_B^* p_A - i \omega \kappa^* p_B^* p_A - q_B^* p_A. \]  

(93)

Since \((83)_A=(83)_B=(84)_A=(84)_B=0\) by virtue of eq. (83) and (84), it follows that the expression at the left-hand side of (89) equals 0 as anticipated. After substituting (90)-(93) into (89),

\[
\int_V d^3r \left( \nabla p_A \cdot \mathbf{v}_B^* + \nabla \cdot \mathbf{v}_B^* p_A \right) + \int_V d^3r \left( \nabla p_B^* \cdot \mathbf{v}_A + \nabla \cdot \mathbf{v}_A p_B^* \right) \\
- \int_V d^3r i \omega (\kappa^* - \kappa) p_A p_B^* - \int_V d^3r (q_A p_B^* + q_B p_A) = 0. \]  

(94)

It is convenient to apply the divergence theorem [e.g. Hildebrand, 1976] to the first two integrals at the left hand side of eq. (94), which then simplifies to

\[
\int_{\partial V} d^2r \left( p_A \mathbf{v}_B^* + p_B^* \mathbf{v}_A \right) \cdot \hat{n} + 2i\omega \int_V d^3r \Im(\kappa)p_A p_B^* - \int_V d^3r (q_B^* p_A + q_A p_B^*) = 0, \]  

(95)

where \(\hat{n}\) is the unit vector normal to \(\partial V\), and \(\Im(\kappa)\) denotes the imaginary part of \(\kappa\). Eq. (95) is equivalent to the “reciprocity theorem of the convolution type”, eq. (5) of Wapenaar and Fokkema [2006] or eq. (4) of Snieder [2007].

An equation relating the cross-correlation of ambient signal to Green’s functions can be found from (95) by considering the case \(q_{A,B}(r) = \delta(r - r_{A,B})\), with \(r_{A,B}\) arbitrary locations in \(V\). It follows that \(p_{A,B} = \mathcal{G}(r, r_{A,B})\). Substituting these expressions for \(p\) and \(q\) into (95) and using (83) to eliminate the velocity,

\[
\mathcal{G}(r_B, r_A) + \mathcal{G}^*(r_A, r_B) = \\
= \frac{1}{i\omega} \int_{\partial V} d^2r \frac{1}{\rho} \left[ \mathcal{G}^* (r, r_B) \nabla \mathcal{G}(r, r_A) - \mathcal{G}(r, r_A) \nabla \mathcal{G}^* (r, r_B) \right] \cdot \hat{n} \\
+ 2i\omega \int_V d^3r \Im(\kappa) \mathcal{G}(r, r_A) \mathcal{G}^*(r, r_B) \]  

(96)

[e.g. Wapenaar and Fokkema, 2006; Snieder, 2007; Campillo and Roux, 2014].
Notice that the treatment that lead from eq. (89) to (96) remains valid for heterogeneous \( \kappa \) and \( \rho \); eq. (96) holds for a heterogeneous, attenuating medium that could be bounded or unbounded. It is thus more general than similar eqs. (32), (53), (63), (72), which are only strictly valid if the propagation medium is homogeneous.

Provided that the medium be smooth or homogeneous at and near \( \partial V \), \( \mathcal{G} \) still coincides with \(-i\omega G_{3D}\) on \( \partial V \), with \( G_{3D} \) the homogeneous-medium Green’s function. \( \nabla \mathcal{G} = -i\omega \nabla G_{3D} \)
can then be computed through eq. (201), which implies

\[
\nabla G_{3D}(\mathbf{r}, \omega) = \frac{\mathbf{r}}{r} \left[ \frac{2}{r} G_{3D}(\mathbf{r}, \omega) - \frac{i\omega}{c} G_{3D}(\mathbf{r}, \omega) \right].
\]

(97)

If we make the further assumption that all sources are far from \( \partial V \) (\( r \) is always large), it follows that \( \frac{2}{r} G_{3D} \) can be neglected in eq. (97), while \( \frac{r}{r} \cdot \hat{n} \approx 1 \), so that

\[
\nabla G_{3D}(\mathbf{r}, \omega) \cdot \hat{n} \approx -\frac{i\omega}{c} G_{3D}(\mathbf{r}, \omega),
\]

(98)

and

\[
\nabla \mathcal{G}(\mathbf{r}, \omega) \cdot \hat{n} \approx -\frac{i\omega}{c} \mathcal{G}(\mathbf{r}, \omega)
\]

(99)

can be computed through eq. (201), which implies

\[
\mathcal{G}(\mathbf{r}_B, \mathbf{r}_A) + \mathcal{G}^*(\mathbf{r}_A, \mathbf{r}_B) =
\approx -\frac{2}{\rho c} \int_{\partial V} d^2 \mathbf{r} \left[ \mathcal{G}^*(\mathbf{r}, \mathbf{r}_B) \mathcal{G}(\mathbf{r}, \mathbf{r}_A) \right]
+ 2i\omega \int_V d^3 \mathbf{r} \Im(\kappa) \mathcal{G}(\mathbf{r}, \mathbf{r}_A) \mathcal{G}^*(\mathbf{r}, \mathbf{r}_B).
\]

(100)

Our treatment in sec. (5.2) was limited to non-attenuating media, where \( \Im(\kappa) = 0 \). In this case eq. (100) reduces to

\[
\mathcal{G}(\mathbf{r}_B, \mathbf{r}_A) + \mathcal{G}^*(\mathbf{r}_A, \mathbf{r}_B) \approx -\frac{2}{\rho c} \int_{\partial V} d^2 \mathbf{r} \left[ \mathcal{G}^*(\mathbf{r}, \mathbf{r}_B) \mathcal{G}(\mathbf{r}, \mathbf{r}_A) \right].
\]

(101)

Applying eq. (88), i.e. replacing \( \mathcal{G} = -i\omega G_{3D} \),

\[
\frac{i\rho c}{2\omega} [G_{3D}^*(\mathbf{r}_A, \mathbf{r}_B) - G_{3D}(\mathbf{r}_B, \mathbf{r}_A)] = -\int_{\partial V} d^2 \mathbf{r} \left[ G_{3D}(\mathbf{r}, \mathbf{r}_B) G_{3D}(\mathbf{r}, \mathbf{r}_A) \right],
\]

(102)
and by virtue of the reciprocity $G_{3D}(r_B, r_A) = G_{3D}(r_A, r_B)$, etc.,

$$\frac{\rho c}{\omega} \Im [G_{3D}(r_A, r_B)] = -\int_{\partial V} d^2 r \ [G^*_3 (r, r_B) G_{3D}(r, r_A)].$$  \hspace{1cm} (103)

The right-hand side of eq. (103) is simply $I_S(\omega)$ as defined by eq. (9); since $n_S$ in eq. (51) is arbitrary, eqs. (103) and (51) are equivalent. In a smooth, lossless medium illuminated from all azimuths, the stationary-phase and reciprocity-theorem approaches lead to the same relationship between Green’s function and cross-correlation, establishing, in practice, that the Green’s function between $r_A$ and $r_B$ can be reconstructed from observations as long as the medium is illuminated by a dense distribution of sources covering its boundary $\partial V$ (sec. 5.2).

The more general, reciprocity-theorem-based results (96) and/or (100) apply to attenuating media. $\Im(\kappa) \neq 0$ implies that the volume-integrals at their right-hand sides cannot be neglected; reconstruction of an attenuating medium’s Green’s function from the data requires that the medium be illuminated by sources within $V$ [e.g. Campillo and Roux, 2014].

### 6.2.2 Seismic waves

In a series of articles, Kees Wapenaar and co-workers have applied the above ideas to the case of an elastic medium, where both compressional and shear deformation exist [e.g., Wapenaar, 2004; Wapenaar and Fokkema, 2006; Wapenaar et al., 2006]. Their procedure, based on applying the reciprocity theorem to a pair of states both excited by impulsive point sources, is qualitatively similar to the acoustic-wave formulation of Snieder [2007], illustrated here in sec. 6.2.1. The most complete description of the reciprocity-theorem approach is that of Wapenaar et al. [2006], who allow for medium inhomogeneity and attenuation. Wapenaar et al. [2006] show that, in analogy with eqs. (100)–(103) for the acoustic case, the Green’s function can be reconstructed from noise cross-correlation provided that the medium is illuminated by noise sources densely distributed throughout a volume $V$, where receivers are immersed. If the medium is lossless, sources within $V$ are unnecessary, but illumination from sources distributed throughout the surface that bounds $V$ is still needed.

### 6.3 Normal-mode approach

Lobkis and Weaver [2001] use a normal-mode approach to find an analytical expression for diffuse-field cross-correlation in a bounded medium. Following Snieder et al. [2010], we briefly repeat their treatment for the simpler case of a lossless bounded medium. Normal modes are
defined as the real functions $p_n(r)$ such that $p_n(r)\cos(\omega_n t)$, $p_n(r)\sin(\omega_n t)$ ($n=1,2,3,\ldots,\infty$), with eigenfrequencies $\omega_n$, form a complete set of solutions to the homogenous version of the scalar wave equation (1). Any wave field $p(r,t)$ propagating in the medium under consideration can be written as a linear combination of modes with coefficients $a_n$, $b_n$

$$p(r,t) = \sum_n [a_n p_n(r) \cos(\omega_n t) + b_n p_n(r) \sin(\omega_n t)],$$

(104)

where $\sum_n$ denotes summation over the integer values of $n$ from 1 to infinity.

### 6.3.1 Green’s function as a linear combination of modes

Let us define the Green’s function $G_M$ as the solution of the homogeneous version ($\frac{\partial F}{\partial t} = 0$) of eq. (1), with initial conditions $p = 0$ and $\frac{\partial p}{\partial t} = \delta(r - s)$ (point source at $s$). $G_M(r,t)$ can be written as a linear combination of modes,

$$G_M(r,t) = \sum_k [\alpha_k \cos(\omega_k t) + \beta_k \sin(\omega_k t)] p_k(r).$$

(105)

Substituting into (105) the initial condition on $p$ we find

$$\sum_k \alpha_k p_k(r) = 0;$$

(106)

after differentiating (105) with respect to time, the initial condition on $\frac{\partial p}{\partial t}$ gives

$$\sum_k \omega_k \beta_k p_k(r) = \delta(r - s).$$

(107)

We multiply both sides of eqs. (106) and (107) by the eigenfunction $p_n(r)$ and integrate over $r$. From eq. (107) we find

$$\sum_k \omega_k \beta_k \int_{\mathbb{R}^3} d^3r p_k(r) p_n(r) = \int_{\mathbb{R}^3} d^3r \delta(r - s)p_n(r),$$

(108)

which, after taking advantage of the orthonormality of the modes (left-hand side) and applying the properties of the $\delta$ function (right-hand side), collapses to

$$\beta_n \omega_n = p_n(s).$$

(109)
Eq. (106) likewise reduces to $\alpha_k = 0$, and eq. (105) becomes

$$G_M(r, t) = \sum_n p_n(s)p_n(r)\frac{\sin(\omega_n t)}{\omega_n},$$  \hspace{2cm} (110)$$
valid for $t > 0$, consistent with equation (4) of Lobkis and Weaver [2001] and equation (1) of Snieder et al. [2010].

6.3.2 Ensemble-averaged cross-correlation as a linear combination of modes

Let us write the ambient signals recorded at $R_1$ and $R_2$ as linear combinations of modes with random coefficients $a_n$, $b_n$. Their cross-correlation then reads

$$\int_{-T/2}^{+T/2} d\tau p^*(r_1, \tau)p(r_2, t + \tau) = \sum_{n,k} \left\{ a_k a_n p_k(r_1)p_n(r_2) \int_{-T/2}^{+T/2} d\tau \cos(\omega_k \tau) \cos[\omega_n(t + \tau)] \\
+ a_k b_n p_k(r_1)p_n(r_2) \int_{-T/2}^{+T/2} d\tau \cos(\omega_k \tau) \sin[\omega_n(t + \tau)] \\
+ b_k a_n p_k(r_1)p_n(r_2) \int_{-T/2}^{+T/2} d\tau \sin(\omega_k \tau) \cos[\omega_n(t + \tau)] \\
+ b_k b_n p_k(r_1)p_n(r_2) \int_{-T/2}^{+T/2} d\tau \sin(\omega_k \tau) \sin[\omega_n(t + \tau)] \right\}$$  \hspace{2cm} (111)$$

Lobkis and Weaver [2001] make the assumption that “modal amplitudes are uncorrelated random variables”, which is equivalent to noise sources being spatially and temporally uncorrelated so that “cross-terms” can be neglected (App. D). This means in practice that if one repeats the cross-correlation (111) at many different times, the normal modes of the medium stay the same (medium properties do not change), but the coefficients $a_n$, $b_n$ change in a random fashion at each realization. When the average of all realizations is taken, the products $a_k a_n$ and $b_k b_n$ both average to $\delta_{kn}M(\omega_n)$, with the function $M$ indicating how strongly different eigenfrequencies are excited on average, and $\delta_{kn} = 1$ if $k = n$, 0 otherwise; $a_k b_n$, on the other hand, averages to 0 for all values of $k, n$. Using $\langle \bullet \rangle$ to denote the averaging.
procedure, it follows from (111) that

\[
\left\langle \int_{-T/2}^{+T/2} d\tau p^*(r_1, \tau)p(r_2, t+\tau) \right\rangle = \sum_{n,k} \left\{ \langle a_k a_n \rangle p_k(r_1)p_n(r_2) \int_{-T/2}^{+T/2} d\tau \cos(\omega_k \tau) \cos[\omega_n(t+\tau)] \right. \\
+ \langle a_k b_n \rangle p_k(r_1)p_n(r_2) \int_{-T/2}^{+T/2} d\tau \cos(\omega_k \tau) \sin[\omega_n(t+\tau)] \\
+ \langle b_k a_n \rangle p_k(r_1)p_n(r_2) \int_{-T/2}^{+T/2} d\tau \sin(\omega_k \tau) \cos[\omega_n(t+\tau)] \\
+ \langle b_k b_n \rangle p_k(r_1)p_n(r_2) \int_{-T/2}^{+T/2} d\tau \sin(\omega_k \tau) \sin[\omega_n(t+\tau)] \left. \right\} \\
= \sum_n M(\omega_n)p_n(r_1)p_n(r_2) \int_{-T/2}^{+T/2} d\tau \cos(\omega_n t) \\
= \sum_n M(\omega_n)p_n(r_1)p_n(r_2)T \cos(\omega_n t), \tag{112}
\]

where the trigonometric identity \( \cos(\omega_n \tau) \cos[\omega_n(t+\tau)] + \sin(\omega_n \tau) \sin[\omega_n(t+\tau)] = \cos(\omega_n t) \) is used [Snieder et al., 2010]. Eq. (111) is equivalent to eqs. (9) or (10) of Snieder et al. [2010], except that we have chosen not to normalize the coefficients \( a_n \), \( b_n \) by the corresponding eigenfrequency \( \omega_n \).

Provided that \( M \) is constant with respect to \( \omega \) (all modes are equally excited), the time-derivative of the right-hand side of eq. (112) is proportional to the right-hand side of (110): in other words, the ensemble-averaged cross-correlation of a diffuse field recorded at \( r_1 \) and \( r_2 \) is proportional to the time-derivative of the Green’s function \( G_M \) (eq. (110) in sec. 6.3.1), for a source located at \( r_1 \) and a receiver located at \( r_2 \). This is equivalent to eqs. (50) and (103), valid for non-steady-state 3-D acoustic media.

### 6.4 Analogy between diffuse field and time-reversal mirror

Derode et al. [2003] explain the relationship between diffuse-field cross-correlation and Green’s function via the concept of time-reversal mirror [e.g., Fink, 1999, 2006]. A time-reversal mirror can be thought of as an array of transducers that record sound, reverse it with respect to time, and emit the time-reversed acoustic signal; if the array is sufficiently large and dense, it will time-reverse the entire propagating wave field, focusing time-reversed waves back to the origin of the initial signal.

Following Stehly [2007], we next summarize the reasoning of Derode et al. [2003] in three simple steps. First of all (sec. 6.4.1), cross-correlating two signals is equivalent to time-reversing the first signal, then convolving it with the second. We then show (sec. 6.4.2) that the convolution of two impulsive signals emitted from \( r \) and recorded at the locations...
and \( r_2 \) coincides with the signal recorded at \( r_2 \) after being emitted by a source at \( r_1 \) and then time-reversed and re-emitted by a transducer at \( r \). It follows (sec. 6.4.3) that, if instead of a single transducer at \( r \) an entire array of transducers forming a time-reversal mirror are present, the mentioned convolution coincides with the Green’s function: through the relationship between convolution and cross-correlation, an equation connecting Green’s function and cross-correlation is thus determined.

### 6.4.1 Convolution, cross-correlation and time reversal

Let us first recall the definition of the convolution \( f \otimes g \) of two real-valued functions \( f(t), g(t) \):

\[
f \otimes g(t) = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau) d\tau.
\]

(113)

It can be shown that the convolution operator is commutative,

\[
f \otimes g = g \otimes f,
\]

(114)

and associative,

\[(f \otimes g) \otimes h = f \otimes (g \otimes h).
\]

(115)

Convolution and cross-correlation are closely related: if one denotes \( f^-(t) \equiv f(-t) \) the function found by systematically changing the sign of the argument of \( f \), that is to say, by time-reversing \( f \), it follows that

\[
f^- \otimes g(t) = \int_{-\infty}^{+\infty} f(-\tau)g(t-\tau) d\tau
\]

\[
= -\int_{-\infty}^{+\infty} f(\tau)g(t+\tau) d\tau
\]

\[
= \int_{-\infty}^{+\infty} f(\tau)g(t+\tau) d\tau,
\]

(116)

i.e., the convolution of \( f^- \) with \( g \) coincides with the cross-correlation of \( f \) with \( g \) [e.g. Smith, 2011].

Now consider a source at the location \( r \) emitting the (real-valued) signal \( e(t) \), and two receivers at the locations \( r_1 \) and \( r_2 \). By definition of Green’s function \( G \), the receiver at \( r_1 \) records a signal \( e \otimes G(r_1, r)(t) \), while that at \( r_2 \) records \( e \otimes G(r_2, r)(t) \) (Fig. 8). Let us cross-correlate the signal recorded at \( r_1 \) with that recorded at \( r_2 \), and denote \( C_{12} \) the cross correlation. Making use of equation (116) and of the fact that convolution is commutative
Figure 8: Setup of sec. 6.4.1, where a signal $e(t)$ emitted by a source at $r$ (star) is recorded by receivers at $r_1$ and $r_2$ (triangles).

and associative,

$$C_{12} = \left[ e^- \otimes G^-(r_1, r) (t) \right] \otimes \left[ e \otimes G(r_2, r) (t) \right] = \left[ G^-(r_1, r) \otimes G(r_2, r) (t) \right] \otimes \left[ e^- \otimes e(t) \right]. \quad (117)$$

### 6.4.2 Propagation from one source to a transducer and from the transducer to a receiver

Let us now place at the location $r_1$ a sound source that emits an impulsive signal, and at $r$ a transducer that records sound, time-reverses it and emits it back (Fig. 8). A receiver is still placed at $r_2$. The signal recorded at $r$ coincides with the Green’s function $G(r, r_1, t)$. The signal $p(r_2, t)$ recorded at $r_2$ at a time $t$ is the convolution of the signal emitted by $r$, that is, the time-reversed Green’s function $G(r, r_1, -t)$, with the Green’s function $G(r_2, r, t)$,

$$p(r_2, t) = G^-(r, r_1) \otimes G(r_2, r) (t). \quad (118)$$

(This is valid in the assumption that the emitted wavelet is short enough, and/or $r$ is far enough from both $r_1$ and $r_2$ for the the time-reversed wave packet $p(r_2, t)$ to be easily isolated from the “direct” arrival at $r_2$.) By the reciprocity of $G$, $G^-(r, r_1, t) = G^-(r_1, r, t)$. We can thus substitute eq. (118) into (117), and

$$C_{12} = p(r_2, t) \otimes \left[ e^- \otimes e(t) \right], \quad (119)$$
Figure 9: Setup of sec. 6.4.2, where an impulse emitted by a source at $r_1$ (triangle) is recorded by a transducer at $r$ (star), which then time-reverses it and emits it back. A receiver at $r_2$ (triangle) would then record the convolution of the time-reversed initial impulse with the Green’s function corresponding to the locations $r$ and $r_2$.

i.e., the cross-correlation of the recordings, made at $r_1$ and $r_2$, of the signal emitted by a source at $r$ coincides with the recording made at $r_2$ of the same signal, emitted at $r_1$ and recorded and time-reversed at $r$. For impulsive signals $e(t) = \delta(t)$, eq. (119) collapses to

$$C_{12} = p(r_2, t).$$

6.4.3 Multiple sources and time-reversal mirror

We next consider a dense, uniform distribution of transducers surrounding the locations $r_1$ and $r_2$. Such a set of transducers forms a time-reversal mirror [e.g., Fink, 1999, 2006]. Equations (117) and (118) remain valid, after replacing $G(r_1, r, t)$, $G(r_2, r, t)$ with the sum of Green’s functions associated with all locations $r$ where transducers are now placed; at the limit of a continuous source distribution,

$$C_{12} = \int_\Omega dr \left[ G^-(r_1, r) \otimes G(r_2, r)(t) \right] \otimes \left[ e^- \otimes e(t) \right],$$

where $\Omega$ is the entire solid angle.

In this setup (Fig. 10), an impulsive signal emitted at $r_1$ is first recorded at $r_2$ as $G(r_2, r_1)$; it then hits the transducer array, which, by definition of time-reversal mirror, sends the signal back in such a way that the same wave field propagates backwards in time: the receiver at
Figure 10: Setups of sec. 6.4.3. (a) Signals emitted by a circle of sources (stars) are recorded by a receiver pair at $r_1$ and $r_2$ (triangles); this is the same setup as in Fig. 8 (sec. 6.4.1), but now there is more than one source. (b) An impulse emitted by a source at $r_1$ is recorded by a set of transducers which time-reverse it and emit it back. This is the same setup at in Fig. 9 (sec. 6.4.2), but with more than one transducer. The transducers occupy the same locations (stars) as the sources in (a) and thus form a time-reversal mirror (in two dimensions). Derode et al. [2003] and Stehly et al. [2006] use the properties of acoustic time reversal to prove that, in a setup such as (a), the cross-correlation between the signals recorded at $r_1$ and $r_2$ can be associated with the Green’s function between the same two locations.

$r_2$ records the time-reversed Green’s function $G(r_2, r_1, -t)$, and the back-propagated signal eventually focuses back on $r_1$ in the form of an impulse at time $-t = 0$.

This means, in practice, that if the individual transducer at $r$ (Fig. 8) is replaced by a time-reversal mirror, then $p(r_2, t)$ in eq. (120) can be replaced by the Green’s function $G(r_2, r_1, -t)$,

$$C_{12} = G(r_2, r_1, -t).$$  \hspace{1cm} (122)

Now recall from sec. 6.4.1 that $C_{12}$ is also the cross-correlation of the recordings made at $r_1$ and $r_2$ of signal generated at $r$; the individual source at $r$ must now be replaced by a set of sources occupying the transducer locations, so that cross-correlations associated with individual sources are summed. Derode et al. [2003] correctly infer that, if a signal is generated by a distribution of sources with the geometry of an effective time-reversal mirror (i.e. energy propagating along all azimuths, diffuse field), the Green’s function between any two points $r_1$ and $r_2$ can be found by cross-correlating the recordings of said signal made at $r_1$ and $r_2$. This statement is, again, equivalent to eqs. (50) and (103).
Figure 11: Distribution of sources (stars) and receivers (triangles) in the setup of sec. 7. Only 10% of the simulated sources are shown. In some of the simulations discussed in the following, the effects of uneven source distributions including only the sources denoted by green vs. yellow stars are modeled.

7 Uneven source distributions and “spurious arrivals”

Our derivation so far is based on the hypothesis that the geographic distribution of noise sources be close to uniform with respect to source-receiver azimuth. The stationary-phase formulae of appendix A only hold if $f$ is a smooth function of $x$ in eq. (150), and of both $x$ and $y$ in eq. (152); the source distributions $n_C$, $n_M$, $n_S$ must accordingly be smooth with respect to $\varphi$ and/or $\theta$ for the treatment of sec. 5 to be valid. The integral in eq. (103) likewise extends to the whole boundary of the volume $V$ containing the receivers: if $\partial V$ is not covered densely and uniformly by sources, noise cross-correlation does not coincide with the right-hand side of eq. (103), and $G$ is not properly reconstructed.

Noise sources are generally not uniformly distributed in practical applications, and we know e.g. from Mulargia [2012] that seismic ambient noise on Earth is not strictly diffuse. We illustrate the consequences of significant inhomogeneities in source distribution with a simple model. As in secs. 4.3 and 5.3, receivers $R1$ and $R2$, lying 20km from one another on a membrane of infinite extension, are surrounded by a circle of sources whose center is $R1$ and whose radius is 100km (Fig. 11). We numerically convolve a Ricker wavelet (central frequency of 1Hz) with the Green’s function $G_{2D}$ for each of the sources in question. Using a wavelet rather than an impulse allows to better visualize the effects we are interested in.
For each location of the source, we cross-correlate the corresponding signals at \( R_1 \) and \( R_2 \) and plot the cross-correlations in Fig. 12a. The result of stacking the cross-correlations, shown in Fig. 12b, is consistent with the results of sec. 5.3, after modulating the Green’s function with the Ricker wavelet (we shall speak of “Ricker response” instead of Green’s function). We next average only the cross-correlations associated with sources denoted in green in Fig. 11, and, finally, only those associated with the “yellow” sources of Fig. 11. Two inferences can be made from Fig. 12c, where both averages are shown: (i) if only the “yellow” sources are “on”, and energy only travels in the direction \( R_2 \rightarrow R_1 \), only the anticausal Ricker’s response between \( R_1 \) and \( R_2 \) emerges from averaging; likewise, only the causal part shows up if only sources to the left of \( R_1 \) are active. (ii) While both causal and anti-causal arrivals in Fig. 12b approximately coincide with those of Fig. 12c, the curves in Fig. 12c contain two additional arrivals, corresponding to the two azimuths where both source distributions in Fig. 11 abruptly end. These arrivals, usually referred to as “spurious”, have no relation to the Ricker response; they are artifacts caused by strong inhomogeneities in the source distribution. Spurious arrivals are likely to affect field data, and can be identified in laboratory (physical acoustics) data.

8 Ambient-signal cross-correlation in the presence of a scatterer: a stationary-phase derivation

The stationary-phase derivations carried out above have established mathematical relationships between two-station ambient-signal cross-correlation and a medium’s Green’s function, in the simple case of homogeneous, unbounded media. The same approach can also usefully be applied to a homogeneous medium including a limited number of point scatterers. Following Snieder et al. [2008], we shall treat in some detail the case of a homogeneous, 3-D acoustic medium containing a single point scatterer: extension to more scatterers [Fleury et al., 2010] is then straightforward, albeit cumbersome.

It is convenient to place the origin of the coordinate system at the location of the scatterer, and to choose the \( x \)- and \( z \)-axes so that the plane they identify contains the locations \( r_1, r_2 \) of receivers \( R_1 \) and \( R_2 \). In this setup, the Green’s function \( G_{3D}^S \) is the sum of \( G_{3D} \) from eq. (201) plus an additional, “scattered” term involving propagation from the source (located at
Figure 12: Cross-correlations associated with a circular, planar distribution of sources surrounding the two receivers as sketched in Fig. 13. Each source generates a Ricker wavelet (central frequency=1Hz); the wavelet is convolved with the Green’s function (phase velocity $c=2$Km/s) to evaluate the signals observed at the two receivers, which are then cross-correlated. (a) Single-source cross-correlations for all source azimuths, the dashed lines mark the azimuths separating the two (yellow vs. green) subsets of sources as defined in Fig. 11. (b) Stacked cross-correlation resulting from (a). (c) Stacked cross-correlations that one would obtain if signal was generated only at the locations identified by green stars in Fig. 11 (green line) vs. the yellow stars (yellow line).
a point $s$) to the scatterer and from the scatterer to the receiver,

$$G^S_{3D}(r_{1,2}, s, \omega) = G_{3D}(r_{1,2S}, \omega) + G_{3D}(r_{1,2}, \omega)G_{3D}(s, \omega)h(\hat{r}_{1,2}, \hat{s}),$$  \hspace{1cm} (123)$$

where $r_{1,2}$ and $s$ denote the moduli of $r_{1,2}$ and $s$, respectively, and $\hat{r}_{1,2}$, $\hat{s}$ are unit-vectors parallel to $r_{1,2}$ and $s$. The scattering function (or “matrix”) $h(\hat{r}_{1,2}, \hat{s})$ accounts for the azimuth-dependence of the amount of scattered energy \cite[e.g.][]{Snieder1996, GroenenboomSnieder1995, Marston2001}.

The frequency-domain cross-correlation of the signals recorded at $R1$ and $R2$ reads

$$G^S_{3D}(r_{1, s}, \omega)G^{S\dagger}_{3D}(r_{2, s}, \omega) = G_{3D}(r_{1S}, \omega)G^{s\dagger}_{3D}(r_{2S}, \omega) + G_{3D}(r_{1S}, \omega)G^{s\dagger}_{3D}(r_{2}, \omega)G_{3D}(s, \omega)h(\hat{r}_{2}, \hat{s})$$

$$+ G_{3D}(r_{1}, \omega)G_{3D}(s, \omega)G^{s\dagger}_{3D}(r_{2S}, \omega)h(\hat{r}_{1}, \hat{s})$$

$$+ G_{3D}(r_{1}, \omega)G_{3D}(s, \omega)G^{s\dagger}_{3D}(r_{2}, \omega)G_{3D}(s, \omega)h(\hat{r}_{1}, \hat{s})h^{\dagger}(\hat{r}_{2}, \hat{s}).$$  \hspace{1cm} (124)$$

We next assume sources to be distributed over a spherical surface of radius $s=R$ surrounding the receivers, as in sec. 3, scenario (ii); for the sake of simplicity, we limit ourselves to uniform source density $n_S(\theta, \varphi) = 1$. Neglecting cross-terms (appendix D), the cumulative effect of such a source distribution is obtained by integrating eq. (124) in $s$ over the whole solid angle. The integral $I_S(\omega)$ of the first term at the right-hand side of (124) has been treated in detail in sec. 5.2 and its analytical form is given, e.g., by eq. (50); substituting $n_S(\theta, \varphi) = 1$,

$$I_S(\omega) \approx \frac{\sqrt{2\pi}}{(4\pi R)^2} \int \frac{d\theta \sin \theta}{\omega} [G_{3D}(\Delta, \omega) - G^{s\dagger}_{3D}(\Delta, \omega)],$$  \hspace{1cm} (125)$$

where $\Delta$ denotes, as usual, the distance between $R1$ and $R2$. Let us call $I_2(\omega)$, $I_3(\omega)$ and $I_4(\omega)$ the source-averages of the remaining three terms at the right-hand side of (124). The first of these integrals,

$$I_2(\omega) = \frac{1}{4\pi} \int_{0}^{\pi} d\theta \sin \theta \int_{-\pi}^{\pi} d\varphi \frac{G_{3D}(r_{1S}(\theta, \varphi), \omega)G^{s\dagger}_{3D}(r_{2}, \omega)G_{3D}(s(\theta, \varphi), \omega)h(\hat{r}_{2}, \hat{s}(\theta, \varphi))}{\omega}$$

$$= \frac{1}{(4\pi)^3} \int_{0}^{\pi} d\theta \sin \theta \int_{-\pi}^{\pi} d\varphi \frac{G^{s\dagger}_{3D}(r_{2} - r_{1S}(\theta, \varphi))}{R r_{2} r_{1S}(\theta, \varphi)} h(\hat{r}_{2}, \hat{s}(\theta, \varphi)),$$  \hspace{1cm} (126)$$

can be simplified by making the hypothesis that the source be very far from both receivers.
and from the scatterer [Snieder et al., 2008], i.e. $R \gg r_{1,2}$, which implies that $r_{1,2S}$ can be
replaced with $R$ at the denominator of (126). At the exponent of the numerator care must be
taken to evaluate the difference $R - r_{1S}$ which is of the same order as $r_2$; in the 3-D Cartesian
reference frame

$$s - r_1 = (R \sin \theta \cos \varphi - r_1 \sin \theta_1, R \sin \theta \sin \varphi, R \cos \theta - r_1 \cos \theta_1), \quad (127)$$

and consequently

$$r_{1S} = |s - r_1| = \left[ R^2 + r_1^2 - 2Rr_1(\sin \theta \cos \varphi \sin \theta_1 + \cos \theta \cos \theta_1) \right]^{\frac{1}{2}}. \quad (128)$$

Eq. (128) implies that

$$r_{1S} \approx R - r_1(\sin \theta \cos \varphi \sin \theta_1 + \cos \theta \cos \theta_1) \quad (129)$$
to first order in $r_1/R$ [Snieder et al., 2008]. Substituting into eq. (126),

$$I_2(\omega) = \frac{1}{(4\pi)^{\frac{3}{2}}} \frac{1}{(c\sqrt{2\pi})^3} \frac{e^{i\varphi_2 r_2}}{R^2 r_2} \int_0^\pi d\theta \sin \theta \int_{-\pi}^\pi d\varphi e^{i\varphi_1} r_1(\sin \theta \cos \varphi \sin \theta_1 + \cos \theta \cos \theta_1) h^*(\hat{r}_2, \hat{s}(\theta, \varphi)). \quad (130)$$

The integral at the right-hand side of eq. (130) coincides with that in (152), after replacing

$\lambda=\omega$, $x=\theta$, $y=\varphi$, $f(\theta, \varphi)=h^*(\hat{r}_2, \hat{s}(\theta, \varphi)) \sin \theta$ and $\psi(\theta, \varphi)=r_1(\sin \theta \cos \varphi \sin \theta_1 + \cos \theta \cos \theta_1)/c$;

the stationary-phase formula (151) can then be applied, provided that $h$ be a smooth function
of $\theta$, $\varphi$.

In analogy with the procedure of sec. 5.2, we differentiate $\psi(\theta, \varphi)$ with respect to $\theta$ and
$\varphi$ to find the stationary points of (130),

$$\psi_\theta = \frac{r_1}{c}(\cos \theta \cos \varphi \sin \theta_1 - \sin \theta \cos \theta_1), \quad (131)$$

$$\psi_\varphi = -\frac{r_1}{c} \sin \theta \sin \varphi \sin \theta_1. \quad (132)$$

Equation (132) establishes that stationary points can be found at either $\varphi=0$ or $\varphi=\pi$, i.e. on
the plane where the scatterer and both receivers are. Substituting $\cos \varphi=\pm 1$ into eq. (131)
we further identify the two stationary points ($\theta=\theta_1, \varphi=0$) and ($\theta=\pi-\theta_1, \varphi=\pi$). If we continue
differentiating,

$$\psi_{\theta\theta} = -\frac{r_1}{c}(\sin \theta \cos \varphi \sin \theta_1 + \cos \theta \cos \theta_1),$$

(133)
\[
\psi_{\varphi \varphi} = -\frac{r_1}{c} \sin \theta \cos \varphi \sin \theta_1, \quad (134)
\]
\[
\psi_{\theta \varphi} = \frac{r_1}{c} \sin \theta \sin \varphi \sin \theta_1. \quad (135)
\]

At the stationary points,
\[
\psi(\theta_1, 0) = \frac{r_1}{c}, \quad (136)
\]
\[
\psi(\pi - \theta_1, \pi) = -\frac{r_1}{c}, \quad (137)
\]
\[
\psi_{\theta \theta}(\theta_1, 0) = -\frac{r_1}{c}, \quad (138)
\]
\[
\psi_{\theta \theta}(\pi - \theta_1, \pi) = +\frac{r_1}{c}, \quad (139)
\]
\[
\psi_{\varphi \varphi}(\theta_1, 0) = -\frac{r_1}{c} \sin^2 \theta_1, \quad (140)
\]
\[
\psi_{\varphi \varphi}(\pi - \theta_1, \pi) = \frac{r_1}{c} \sin^2 \theta_1, \quad (141)
\]
\[
\psi_{\theta \varphi}(\theta_1, 0) = \psi_{\theta \varphi}(\pi - \theta_1, \pi) = 0. \quad (142)
\]

After substituting (136)-(142) into the stationary-phase formula (161),
\[
I_2(\omega) \approx \frac{1}{(4\pi)^4} \frac{1}{\sqrt{2\pi c^2 R^2}} \frac{e^{i\frac{\pi}{2} r_2}}{r_2} \frac{i}{\omega} \left[ h^*(\hat{r}_2, \hat{s}(\pi - \theta_1, \pi)) e^{-i\frac{\pi}{2} r_1} \frac{e^{i\frac{\pi}{2} r_1}}{r_1} - h^*(\hat{r}_2, \hat{s}(\theta_1, 0)) h^*(\hat{r}_1, \hat{s}(\theta_1, 0)) \right]. \quad (143)
\]

The integral in \( I_3(\omega) \) is similar to that in \( I_2(\omega) \), and the stationary-phase approximation leads to
\[
I_3(\omega) \approx \frac{1}{(4\pi)^4} \frac{1}{\sqrt{2\pi c^2 R^2}} \frac{e^{-i\frac{\pi}{2} r_1}}{r_1} \frac{i}{\omega} \left[ h(\hat{r}_1, \hat{s}(\theta_2, 0)) e^{-i\frac{\pi}{2} r_2} \frac{e^{-i\frac{\pi}{2} r_2}}{r_2} - h(\hat{r}_1, \hat{s}(\pi - \theta_2, \pi)) \right]. \quad (144)
\]

The integral in \( I_4(\omega) \),
\[
I_4(\omega) = \frac{1}{(4\pi)^3} \frac{1}{(2\pi)^2 c^4} \frac{e^{i\frac{\pi}{2} (r_2 - r_1)}}{R^2 r_2 r_1} \int_0^\pi d\theta \sin \theta \int_0^\pi d\varphi h(\hat{r}_1, \hat{s}(\theta, \varphi)) h^*(\hat{r}_2, \hat{s}(\theta, \varphi)), \quad (145)
\]
is not a stationary-phase integral in general.

Notice that, by eq. (201), and since \( \hat{s}(\theta_1, 0) = \hat{r}_2 \), the term
\[
\frac{e^{i\frac{\pi}{2} r_2}}{r_2} \frac{e^{i\frac{\pi}{2} r_1}}{r_1} h^*(\hat{r}_1, \hat{s}(\theta_1, 0)) = 2^{5} \pi^{3} c^{2} G_{3D}(r_2, \omega) G_{3D}(r_1, \omega) h^*(\hat{r}_1, \hat{r}_2), \quad (146)
\]
appearing in (143), describes (the complex conjugate of) an impulse propagating from \( R2 \) to
is an impulse propagating from $R_1$ to $R_2$ via the scatterer. The remaining terms in (143), (144) and (145) do not have an immediate physical explanation. The source-averaged cross-correlation, coinciding with the integral of (124), takes the form

$$I_S(\omega) + I_2(\omega) + I_3(\omega) + I_4(\omega) \approx \frac{\sqrt{2\pi}}{(4\pi R)^2} \frac{i}{\omega} [G_{3D}(\Delta, \omega) - G_{3D}^*(\Delta, \omega)]$$

$$- \frac{\sqrt{2\pi}}{(4\pi R)^2} \frac{i}{\omega} G_{3D}^*(r_2, \omega)G_{3D}^*(r_1, \omega)h^*(\hat{r}_1, \hat{r}_2)$$

$$+ \frac{\sqrt{2\pi}}{(4\pi R)^2} \frac{i}{\omega} G_{3D}^*(r_2, \omega)G_{3D}(r_1, \omega)h(\hat{r}_2, \hat{r}_1)$$

$$+ \frac{1}{(4\pi)^2} \frac{1}{\sqrt{2\pi c^2}} \frac{\omega}{R^2 r_1 r_2}$$

$$\times \left\{ [h^*(\hat{r}_2, -\hat{r}_1) - h(\hat{r}_1, -\hat{r}_2)] - \frac{1}{4\pi c^2} \int_0^{\pi} d\theta \sin \theta \int_{-\pi}^{\pi} d\varphi h(\hat{r}_1, \hat{s}(\theta, \varphi))h^*(\hat{r}_2, \hat{s}(\theta, \varphi)) \right\}$$

(148)

where we have used eq. (49) with $n_S = 1$, and the identities $\hat{s}(\pi - \theta_1, \pi) = -\hat{r}_1$, $\hat{s}(\pi - \theta_2, \pi) = -\hat{r}_2$.

Equation (148) was first derived, using a slightly different convention/notation, by Snieder et al. [2008], who observed that the term in \{ \ldots \} is zero as a consequence of the generalized optical theorem. We are left with

$$I_S(\omega) + I_2(\omega) + I_3(\omega) + I_4(\omega) \approx \frac{\sqrt{2\pi}}{(4\pi R)^2} \frac{i}{\omega} [G_{3D}^S(\Delta, \omega) - G_{3D}^{S*}(\Delta, \omega)]$$

$$\approx -\frac{\sqrt{2\pi}}{8(\pi R)^2} \frac{1}{\omega^3} \left[G_{3D}^S(\Delta, \omega)\right],$$

(149)

similar to eqs. (51) and (103). The procedure of sec. 5.3.3 could be applied to show that, in the time domain, the source-averaged cross-correlation (149) is proportional to the sum of $G_{3D}^S(t)$ (causal part) and $-G_{3D}^S(-t)$ (anticausal).

This result confirms that, as first pointed out by Weaver and Lobkis [2004], diffuse-field cross-correlation in heterogeneous (rather than just homogeneous) media allows in principle to reconstruct the full Green’s function of the medium, with all reflections and scatterings and propagation modes. This is implicit in the reciprocity-theorem formulation of sec. 6.2, and has been verified experimentally and numerically by, e.g., Larose et al. [2006], Mikesell.
et al. [2012] and Colombi et al. [2014].

8.1 “Spurious” arrivals and their cancellation

The result (149) might be surprising if one considers the scatterer as a (“secondary”) source; no matter where the actual (“primary”) source is, the scatterer is always at the same location relative to $R_1$ and $R_2$, so that the delay between the arrival of the scattered signal at $R_1$ and its arrival at $R_2$ is always the same. This would give rise to a peak in the stacked cross-correlation, that does not correspond with either of the two arrivals in $G_{3D}^S$ as defined by eq. (123).

This is illustrated in Figs. 13 and 14. Following Snieder et al. [2008], we convolve a Ricker wavelet (central frequency of 1Hz) with the Green’s function associated with a uniformly dense, circular source distribution (Fig. 13) plus a single scatterer. For simplicity and in analogy with Snieder et al. [2008], we work in two dimensions (membrane waves from point sources) and assume isotropic scattering. We implement the expression for $h$ derived by Groenenboom and Snieder [1995] via the optical theorem [Newton, 2002]. For each location of the source, we cross-correlate the corresponding signals at $R_1$ and $R_2$, which we choose to be equidistant from the scatterer, and plot the cross-correlations in Fig. 14a. As anticipated, whatever the source azimuth, a peak in the cross-correlation appears at $t=0$, corresponding to the scattered signal hitting simultaneously $R_1$ and $R_2$. This peak, or “arrival”, at $t=0
Figure 14: Cross-correlations associated with a circular, planar distribution of sources surrounding the two receivers, and one isotropic point scatterer, as sketched in Fig. 13. Each source generates a Ricker wavelet (central frequency=1Hz); the wavelet is convolved with the Green’s function (phase velocity $c=2$Km/s), including the contribution of scattering, to evaluate the signals observed at the two receivers, which are then cross-correlated. (a) Single-source cross-correlations for all source azimuths; the color scale saturates at 20% of the maximum amplitude to emphasize the spurious arrivals; the dashed lines mark the azimuths where the “yellow”/“green” source distribution is discontinuous (Fig. 13). (b) Source-averaged cross-correlation resulting from (a). (c) Source-average cross-correlations that one would obtain if signal was generated only at the locations identified by green stars in Fig. 13 (green line) vs. the yellow stars (yellow line).
clearly does not exist in $G_{3D}^S$, which would seem to contradict eq. (149). However, when
stacking all single-azimuth cross-correlations, the $t = 0$ peak cancels out with the “knees”
of some other cross-correlation peaks (Fig. 14b), and eq. (149) is indeed confirmed. The
four stationary points identified in sec. 8 appear as knees of the cross-correlation peaks in
Fig. 14a, corresponding in turn to the two causal and two anticausal peaks of the system’s
Ricker response (Fig. 14b).

It is critical that source illumination be uniform, for the spurious term associated with
the scatterer to disappear. To emphasize this point, we show in Fig. 14c the stacks obtained
by considering only half of the available sources [Snieder et al., 2008], on either side of the
receiver array. Depending on which side, only the causal or anticausal part of the Ricker
response is reconstructed, as in Fig. 12. Furthermore, three artifacts emerge; the one at
$t=0$ clearly results from the spurious wavelet in Fig. 14a, which does not cancel out since
the illumination is (strongly) nonuniform. The other two are simply the spurious arrivals of
Fig. 12, which result from the sharp inhomogeneities in source distribution, and would be
found even in the absence of scatterers.

Scattering thus has a complex effect on Green’s function reconstruction. Each scatterer
further complicates the Green’s function, introducing an additional, physical term to be
reconstructed. It also hinders its retrieval, generating a spurious term which will only cancel
out if the wave field is sufficiently diffuse. On the other hand, scatterers themselves contribute
to the wave field’s diffusivity and azimuthal uniformity of illumination. Recent and current
work [e.g. Fleury et al., 2010; Mikesell et al., 2012; Ravasi and Curtis, 2013; Colombi et al.,
2014] aims at disentagling the specific role of scattered ambient signal in reconstructing the
main peaks of the Green’s function as well as its “coda”.

9 Summary

In a diffuse ambient wave field, i.e. a random wave field where energy propagates with equal
probability in all directions, cross-correlating the signal recorded by a pair of receivers is a
way to measure the impulse response (Green’s function) between the receivers. In practice,
long recordings of seismic ambient noise often include waves traveling along approximately
all azimuths: the combination of their cross-correlations approximates that of a diffuse field.
The relationship between cross-correlation and Green’s function changes depending on some
properties of the medium and of the wave field:
(i) In a homogeneous, lossless, unbounded membrane where circular waves are generated by point sources, the cross-correlation of diffuse noise is generally not explicitly related to the corresponding Green’s function $G_{2D}$ (sec. 5.3). If, however, the source distribution is symmetric along the receiver-receiver axis, eq. (70) stipulates that $G_{2D}$ is proportional to the time-derivative of the cross-correlation. The membrane setup is relevant to seismology, because it corresponds in practice to the propagation of Rayleigh waves (sec. 2.3) on Earth.

(ii) In homogeneous, lossless, unbounded 3-D acoustic media, the Green’s function can be exactly reconstructed from diffuse noise generated at point sources that are smoothly distributed over a sphere surrounding the receivers. More precisely, the cross-correlation coincides with the time-derivative of the Green’s function (sec. 5.2). This relationship holds for both causal and anticausal contributions to the Green’s function/cross-correlation, and remains valid if the source distribution is asymmetric (provided that it be smooth).

(iii) The latter result does not hold if noise in free space is generated by point sources uniformly distributed along a circle surrounding the receivers (sec. 5.1). Still, even in this case, the time of maximum cross-correlation clearly coincides with that of $G_{2D}$’s peak (Fig. 5).

(iv) If noise is made up of plane waves propagating in all directions (sec. 5.4), the same formulation holds for 2-D and 3-D unbounded, lossless media. In this case, the source-averaged cross-correlation is proportional, in the frequency domain, to the real part of the 2-D Green’s function $G_{2D}$, which in turn is proportional to the Bessel function $J_0(\omega r/c)$, with $r$ the inter-receiver distance. Hence eq. (72). The corresponding time-domain relationship was derived by Nakahara [2006], who showed that the Hilbert transform of diffuse-field cross-correlation coincides with the (causal minus anticausal) $G_{2D}$.

(v) In bounded, heterogeneous, attenuating media, a general relationship between source-averaged cross-correlation and Green’s function is found via the reciprocity theorem (sec. 6.2). Essentially, source-averaged cross-correlation coincides with the sum of a volume ($V$) integral, and an integral over the boundary $\partial V$ of such volume. It follows that densely distributed noise sources throughout a volume are needed for the Green’s
function to be reconstructed via cross-correlation; sources within $V$ are unnecessary if
the medium is lossless, provided that all azimuths are illuminated.

The results (i) and (iv) are relevant for most seismology applications seen so far, since,
provided that attenuation can be neglected, they apply to surface waves in the “membrane”
approximation [Tanimoto, 1990] (sec. 2.1). E.g. Ekström et al. [2009] have shown explicitly
that the phase of the Green’s function is in agreement with the data, thus validating the
“lossless-medium” approximation for phase/group-velocity-based seismic imaging, while re-
producing the Green’s function’s amplitude remains problematic [e.g., Weemstra et al., 2014].

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### A The stationary phase approximation

### A.1 1-D integrals

The stationary phase approximation [e.g., *Bender and Orszag*, 1978] applies to integrals of the form

\[
I(\lambda) = \int_a^b f(x)e^{i\lambda \psi(x)} dx,
\]

where \(a\) and \(b\) are arbitrary constants, and \(\lambda\) is large enough for \(e^{i\lambda \psi(x)}\) to be a rapidly oscillating function of \(x\), with respect to the smooth functions \(f(x), \psi(x)\). It is based on the finding [e.g., *Bender and Orszag*, 1978, sec. 6.5] that the leading contribution to \(I(\lambda)\) comes from a small interval surrounding the “stationary points” of \(\psi(x)\), i.e. the locations \(x\) such that \(\psi'(x) = 0\) (where the “prime” denotes differentiation with respect to \(x\)). Elsewhere, the integrand oscillates quickly and its average contribution to the integral is negligible. *Bender and Orszag* [1978] demonstrate that, for a single stationary point at \(a\),

\[
I(\lambda) \approx f(a)e^{i(\lambda \psi(a) \pm \frac{\pi}{4})} \left( \frac{\pi}{2\lambda|\psi''(a)|} \right),
\]

valid for \(\lambda \to \infty\) and in the assumption that \(\psi''(a) \neq 0\). The sign of \(\pi/4\) at the exponent of \(e\) is positive if \(\psi''(a) > 0\) and negative otherwise. Since any integral of the form (150) can be written as a sum of integrals with one of the integration limits coinciding with a stationary
point, eq. (151) is sufficient to solve all 1-D stationary phase integrals like (150), regardless of the number and location of stationary points [Bender and Orszag, 1978].

A.2 Extension to 2-D integrals

The result of section A.1 can be used to find a general approximate formula for integrals of the form

\[ I(\lambda) = \int_a^b \int_c^d dy f(x, y) e^{i\lambda \psi(x, y)}. \]  

(152)

If we, again, only consider the limit \( \lambda \rightarrow \infty \), the integrand in (152) turns out to be very strongly oscillatory, and the only nonnegligible contribution to the integral comes from the vicinity of the stationary points \((x_i, y_i)\), defined by

\[ \left( \frac{\partial \psi}{\partial x}(x_i, y_i), \frac{\partial \psi}{\partial y}(x_i, y_i) \right) = (0, 0) \]  

(153)

[e.g., Wong, 1986]. Equation (152) can thus be approximated by

\[ I(\lambda) \approx \sum_i f(x_i, y_i) \int_{x_i-\varepsilon}^{x_i+\varepsilon} dx \int_{y_i-\delta}^{y_i+\delta} dy e^{i\lambda \psi(x, y)}, \]  

(154)

where the sum is extended to all stationary points \(i\), and \(\varepsilon\) and \(\delta\) are small. \(f(x, y)\) is approximated by \(f(x_i, y_i)\) since \(f\) varies much more slowly than \(e^{i\lambda \psi}\) when \(\lambda\) is large. We next conduct a Taylor series expansion of \(\psi(x, y)\) around \((x_i, y_i)\),

\[ \psi(x, y) \approx \psi(x_i, y_i) + \frac{1}{2} \left[ \psi_{xx}(x-x_i)^2 + \psi_{yy}(y-y_i)^2 + 2\psi_{xy}(x-x_i)(y-y_i) \right], \]  

(155)

where \(\psi_{xx}\) stands for \(\frac{\partial^2 \psi}{\partial x^2}(x_i, y_i)\), and so on. Substituting (155) into (154) and after the changes of variable \(u_i = x - x_i\) and \(v_i = y - y_i\), we find

\[ I(\lambda) \approx \sum_i f(x_i, y_i) e^{i\lambda \psi(x_i, y_i)} \int_{-\varepsilon}^{+\varepsilon} du_i \int_{-\delta}^{+\delta} dv_i e^{\frac{i}{2} \left( \psi_{xx} u_i^2 + \psi_{yy} v_i^2 + 2\psi_{xy} u_i v_i \right)}, \]  

(156)

The integral in (156) can be solved analytically by first rewriting

\[ \psi_{xx} u_i^2 + \psi_{yy} v_i^2 + 2\psi_{xy} u_i v_i = \psi_{xx}(u_i + \frac{\psi_{xy}}{\psi_{xx}} v_i)^2 + v_i^2 \left( \psi_{yy} - \frac{\psi_{xy}^2}{\psi_{xx}} \right), \]  

(157)
and substituting into (156),

\[ I(\lambda) \approx \sum_i f(x_i, y_i) e^{i \lambda \psi(x_i, y_i)} \int_{-\delta}^{+\delta} dv_i e^{i \frac{1}{2} \nabla^2 (\psi_{yy} - \frac{\psi^2_{xx}}{\psi_{xx}^2})} \int_{-\varepsilon}^{+\varepsilon} du_i e^{i \frac{1}{2} \psi_{xx}(u_i + \frac{\psi_{yy}}{\psi_{xx}^2} v_i)^2}. \]  

(158)

Since \( \lambda \to \infty \), both integrals at the right-hand side are 1-D stationary phase integrals as seen in sec. A.1. The only stationary point in the \( v_i \)-integration domain is at \( v_i = 0 \); the phase term in the \( u_i \)-integral is likewise stationary at \( u_i = 0 \) (provided that \( v_i \approx 0 \)). Application of eq. (151) to the \( u_i \)-integral in (158) then gives

\[ \int_{-\varepsilon}^{+\varepsilon} du_i e^{i \frac{1}{2} \psi_{xx}(u_i + \frac{\psi_{yy}}{\psi_{xx}^2} v_i)^2} = e^{\pm i \frac{\pi}{4}} \sqrt{\frac{2\pi}{|\psi_{xx}|}}. \]  

(159)

where we have assumed \( v_i \approx 0 \) since this integral is to be evaluated near the stationary point of the \( v_i \)-integral, and we have implicitly multiplied by 2 since (151) is valid for a single stationary point located at one of the integration limits. Similarly,

\[ \int_{-\delta}^{+\delta} dv_i e^{i \frac{1}{2} \nabla^2 (\psi_{yy} - \frac{\psi^2_{xx}}{\psi_{xx}^2})} = e^{\pm i \frac{\pi}{4}} \sqrt{\frac{2\pi}{|\psi_{yy} - \frac{\psi^2_{xx}}{\psi_{xx}^2}|}.} \]  

(160)

Substituting eqs. (159) and (160) into (158), we are left with the final formula

\[ I(\lambda) \approx \frac{2\pi}{\lambda} \sum_i f(x_i, y_i) e^{i \lambda \psi(x_i, y_i) \pm \frac{\pi}{4} \pm \frac{\pi}{4}} \frac{1}{|\psi_{xx}| |\psi_{yy} - \frac{\psi^2_{xx}}{\psi_{xx}^2}|}. \]  

(161)

valid for \( \lambda \to \infty \) and in the assumption that \( \psi_{xx} \neq 0, \psi_{yy} - \frac{\psi^2_{xx}}{\psi_{xx}^2} \neq 0 \). The signs of the \( \pi/4 \) terms in (161) depend on those of \( \psi_{xx} \) and \( \psi_{yy} - \frac{\psi^2_{xx}}{\psi_{xx}^2} \) (see sec. A.1) [Bender and Orszag, 1978].

### B Fourier transform convention

Inverse and forward Fourier transformations are applied frequently throughout this study. We denote \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) the forward and inverse Fourier transform operators, respectively. \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) can be defined in various ways. We adopt here the following convention: for any function \( f \),

\[ f(\omega) = \mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt \ f(t) e^{-i\omega t}, \]  

(162)
and consequently
\[ f(t) = \mathcal{F}^{-1}[f(\omega)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega f(\omega) e^{i\omega t}, \]
(163)
where we have chosen to simply specify the argument (time \( t \) or frequency \( \omega \), respectively) to distinguish a time-domain function \( f(t) \) from its Fourier transform \( f(\omega) \).

Some properties of the Fourier transform are particularly useful in the context of ambient-noise cross-correlation. First of all, it follows from (162) that the Fourier transform of the derivative of \( f \) with respect to \( t \) is
\[ \mathcal{F}\left[ \frac{df(t)}{dt} \right] = i\omega f(\omega). \]
(164)
The Fourier transform of the integral of \( f \) is
\[ \mathcal{F}\left[ \int_{-\infty}^{t} f(\tau)d\tau \right] = -\frac{i}{\omega} f(\omega), \]
(165)
provided that \( f(t) \rightarrow 0 \) when \( t \rightarrow -\infty \).

Other useful properties of the Fourier transform concern even and odd functions. The Green’s functions \( G(t) \) we work with throughout this study (App. E) are real (in the time domain) and have the property \( G(t) = 0 \) if \( t < 0 \). Let us define the real, even function
\[ G_e(t) = \frac{1}{2} G(t) + \frac{1}{2} G(-t), \]
(166)
and the real, odd function
\[ G_o(t) = \frac{1}{2} G(t) - \frac{1}{2} G(-t). \]
(167)
The definitions (166) and (167) imply
\[ G(t) = G_e(t) + G_o(t). \]
(168)
It follows from (162) that the Fourier transform of a real even function is even and purely real, while the Fourier transform of a real odd function is odd and purely imaginary. Then,
\[ \Re [G(\omega)] = G_e(\omega) \]
(169)
and
\[ \Im [G(\omega)] = -iG_o(\omega), \]
(170)
where $\Re[G(\omega)]$ and $\Im[G(\omega)]$ denote the real and imaginary part of $G(\omega)$, respectively.

C  Bessel functions

Bessel functions emerge frequently in noise literature, starting with the early works of, e.g., Eckart [1953], Aki [1957], Cox [1973]. Their mathematical properties are described in detail by Abramowitz and Stegun [1964]. In seismic interferometry we are in practice only interested in 0th-order Bessel functions of the first and second kind, denoted $J_0(\cdot)$, $Y_0(\cdot)$, respectively, which together with the Hankel functions $H_0^{(1)}(\cdot) = J_0(\cdot) + iY_0(\cdot)$ and $H_0^{(2)}(\cdot) = J_0(\cdot) - iY_0(\cdot)$ can be defined as the solutions of the 0th-order Bessel equation

$$\frac{d^2 f(x)}{dx^2} + \frac{1}{x} \frac{df(x)}{dx} + f(x) = 0 \quad \text{(171)}$$

[Abramowitz and Stegun, 1964, eq. (9.1.1)].

In our implementation we employ the far-field (large $r$) and/or high-frequency (large $\omega$) approximations for $J_0(x)$, $Y_0(x)$ and $H_0^{(1)}(x)$, $H_0^{(2)}(x)$, namely

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos \left(x - \frac{\pi}{4}\right), \quad \text{(172)}$$

$$Y_0(x) \approx \sqrt{\frac{2}{\pi x}} \sin \left(x - \frac{\pi}{4}\right), \quad \text{(173)}$$

$$H_0^{(1)}(x) \approx \sqrt{\frac{2}{\pi x}} e^{i(x-\pi/4)}, \quad \text{(174)}$$

$$H_0^{(2)}(x) \approx \sqrt{\frac{2}{\pi x}} e^{-i(x-\pi/4)} \quad \text{(175)}$$

[Abramowitz and Stegun, 1964, eqs. (9.2.1)-(9.2.4)]. In practice, $J_0$ and $Y_0$ can roughly be thought of as “damped” sinusoidal functions, whose amplitude decays exponentially as their argument tends to infinity; at large $x$, $J_0$ approximates a cosine with a $\pi/4$ phase shift, and $Y_0$ approximates a sine with a $\pi/4$ phase shift.

D  Cancellation of “cross terms”

Throughout this study, simple analytical formulae for the cross-correlation of an ambient wave field are obtained neglecting the contribution of the so-called “cross-terms”, i.e. the receiver-receiver cross-correlation of signal generated by a couple of different sources. We
propose in the following a simple proof, similar to that of Weemstra et al. [2014], of the validity of this assumption.

The pressure due to an impulse emitted by source $j$ at a random time $t_j$ is given by equation (200), after replacing $t - x/c$ with $t - t_j - x/c$ as argument of the Dirac function; in the frequency domain, this is equivalent to adding a phase anomaly $-i \omega t_j$ to the argument of the exponential in eq. (201). If $N_S$ such sources are active during the time interval over which a cross-correlation is conducted, the pressure $p_i$ recorded at receiver $i$ is a linear combination of the signals originating from these sources, i.e., in 3-D,

$$p_i(\omega) = \sum_{j=1}^{N_S} \frac{e^{-i\left(\frac{\omega t_j}{c} + \omega t_j\right)}}{\sqrt{2\pi 4\pi c r_{ij}}},$$

where $r_{ij}$ is the distance between receiver $i$ and source $j$.

Cross-correlating the signal recorded at $R_1$ with that recorded at $R_2$, we find

$$p_1(\omega) p_2^*(\omega) = \sum_{j=1}^{N_S} \sum_{k=1}^{N_S} \frac{e^{-i\frac{\pi}{2}(r_{1j} - r_{2k})}}{2\pi (4\pi c)^2 r_{1j} r_{2k}} e^{-i\omega(t_j - t_k)},$$

and the cross-term contribution can be isolated by separating $j \neq k$ terms (cross terms) from $j = k$ ones:

$$p_1(\omega) p_2^*(\omega) = \sum_{j=1}^{N_S} \frac{e^{-i\frac{\pi}{2}(r_{1j} - r_{2j})}}{r_{1j} r_{2j}} + \sum_{j=1}^{N_S} \sum_{j \neq k} \frac{e^{-i\frac{\pi}{2}(r_{1j} - r_{2k})}}{r_{1j} r_{2k}} e^{-i\omega(t_j - t_k)}.$$

The first term at the right-hand side of eq. (178) is a sum of single-source cross-correlations in 3-D (eq. (6)). (The algebra is similar in 2-D (eq. (7)), if the far field approximation is applied.) Its average is given e.g. by equations eqs. (8)-(10) above, which are obtained from (6) or (7) by neglecting the cross terms $j \neq k$ and replacing the sum over sources $j$ with an integral over the area or volume occupied by the sources.

While $j \neq k$ terms in (178) are non-negligible, we next show that their contribution to the average of (178) over a large set of sources is negligible. We make the assumptions that (i) at each realization, the values $t_j$ change randomly, and (ii) they are uniformly distributed between 0 and $T_a$, i.e., $0 < t_j < T_a$, where $T_a = 2\pi/\omega$. Let us introduce the phase $\phi_j = \omega t_j$, which is randomly distributed between 0 and $2\pi$. The exponent $-i\omega(t_j - t_k)$ is accordingly replaced by $-i(\phi_j - \phi_k)$. In the process of averaging, impulses will be generated multiple times at each source location $j$, resulting in random phases $\phi_j$. This is equivalent to requiring, as it
is usually done in the literature, that noise sources be spatially and temporally uncorrelated [e.g., Snieder, 2004; Roux et al., 2005]. To take into account the effects of random variations in phase, the \( j \neq k \) term must consequently involve (besides the usual sum or integral over sources) an integral over all possible values (0 to \( 2\pi \)) of each source phase,

\[
\begin{align*}
\mathbb{A}v & \left[ \sum_{j=1}^{N_S} \sum_{j \neq k} \frac{e^{i \frac{\pi}{c} (r_{1j} - r_{2k})}}{2 \pi r_{1j} r_{2k}} e^{i (\phi_j - \phi_k)} \right] = \frac{1}{(2\pi)^{N_S}} \int \int \cdots \int_0^{2\pi} \sum_{j=1}^{N_S} \sum_{j \neq k} \frac{e^{i \frac{\pi}{c} (r_{1j} - r_{2k})}}{2 \pi r_{1j} r_{2k}} e^{i (\phi_j - \phi_k)} d\phi_1 d\phi_2 \cdots d\phi_{N_S}.
\end{align*}
\]

(179)

All Integrals evaluate to zero because the integrands traverse a circle in the complex plane from 0 to \( 2\pi \). Consequently, if cross-correlations are averaged over sufficient realizations, the cross-correlation in eq. (178) reduces to a sum of single-source cross-correlations (non-cross-terms).

The averaging procedure we just described is often referred to as “ensemble-averaging” in ambient-noise theory. This expression is borrowed from statistical mechanics: an “ensemble” is a set of states of a system, each described by the same set of microscopic forces and sharing some common macroscopic property. The ensemble concept then states that macroscopic observables can be calculated by performing averages over the states in the ensemble [Tuckerman, 1987]. In our case, one state consists of the same acoustic or elastic medium (its response to a given impulse is always the same) being illuminated by one or more randomly located sources with random phases: our ensemble average is the average over all possible combination of source locations and phases. Cross-correlating recordings associated with a single state (a unique combination of sources) yields many cross-terms; we have just shown, however, that ensemble-averaging cross-correlations over different states implies that these cross-terms stack incoherently and hence become negligible. In the real world, the location and phase of noise sources (e.g., ocean microseisms) are not well known; the assumption is made that, over time, a sufficiently diverse range of sources is sampled, so that averaging the cross-correlation over time (i.e., computing the cross-correlation over a very long time window) is practically equivalent to ensemble-averaging.
E  Green’s functions of the scalar wave equation (homogeneous lossless media)

E.1  Green’s problem as homogeneous equation

Following e.g. Roux et al. [2005], Sanchez-Sesma and Campillo [2006], Harmon et al. [2008], we call Green’s function \( G = G(x, x_S, t) \) the solution of

\[
\nabla^2 G - \frac{1}{c^2} \frac{\partial^2 G}{\partial t^2} = 0 \tag{180}
\]

with initial conditions

\[
G(x, x_S, 0) = 0, \quad \frac{\partial G}{\partial t}(x, x_S, 0) = \delta(x - x_S), \tag{181}
\]

i.e. an impulsive source at \( x_S \). We are only interested in causal Green’s functions, satisfying the radiation condition, i.e. vanishing at \( t < 0 \).

Once \( G \) is known, it can be used to solve rapidly more general initial-value problems associated with (180). Consider for example

\[
\nabla^2 f - \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = 0 \tag{183}
\]

with the more general initial conditions

\[
f(x, 0) = 0, \quad \frac{\partial f}{\partial t}(x, 0) = h(x). \tag{184}
\]

It can be proved by direct substitution that if \( G \) solves (180)-(182) then

\[
f(x, t) = \int_{\mathbb{R}^d} d^d x' G(x, x', t) h(x') \tag{186}
\]

solves (183)-(185). \( d \) in eq. (186) denotes the number of dimensions: 2 or 3 in our case.

For the sake of simplicity, we shall set \( x_S = 0 \) in the following. \( G(x, x_S, t) \) can be recovered from \( G(x, 0, t) \) by translation of the reference frame.
E.2 Solution in the spatial-Fourier-transform domain

Equation (180) can be solved via a spatial Fourier transform, i.e. by the Ansatz

\[ G(x, t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d^d k \, G(k, t) e^{ik \cdot x}, \quad (187) \]

which, substituted into (180), gives

\[ \frac{1}{c^2} \frac{\partial^2 G}{\partial t^2}(k, t) + k^2 G(k, t) = 0, \quad (188) \]

with \( k = |k| \). Eq. (188) is solved by

\[ G(k, t) = A(k) \cos(kct) + B(k) \sin(kct), \quad (189) \]

where \( A(k) \) and \( B(k) \) must be determined by the initial conditions. Transforming (181) and (182) to \( k \)-space, with

\[ \delta(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d^d k \, e^{ik \cdot x}, \quad (190) \]

and replacing \( G \) in the resulting equations with its expression (189), we find \( A(k) = 0, \)

\( B(k) = \frac{1}{kc}, \)

and

\[ G(k, t) = \frac{1}{kc} \sin(kct). \quad (191) \]

Eq. (187) can now be used to determine \( G(x, t) \) from its spatial Fourier transform \( G(k, t) \), and the result differs importantly depending on \( d \).

E.3 Inverse transform of the solution to 2-D space

Substituting (191) into (187) in the \( d = 2 \) case,

\[ G_{2D}(x, t) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \frac{\sin(kct)}{kc} e^{ik \cdot x}. \quad (192) \]

We call \( \xi \) the angle between \( k \) and \( x \). It follows that \( k \cdot x = kx \cos \xi \), with \( x = |x| \). For each \( x \) the integration can be conducted over \( k \) and \( \xi \), using \( dk_1 dk_2 = kdkd\xi \),

\[ G_{2D}(x, t) \]

\[ = \frac{1}{4\pi^2 c} \int_{0}^{+\infty} dk \sin(kct) \int_{0}^{2\pi} d\xi \, e^{ikx \cos \xi} \]

\[ = \frac{1}{2\pi^2 c} \int_{0}^{+\infty} dk \sin(kct) \int_{0}^{\pi} d\xi \, \cos(kx \cos \xi) \]

\[ = \frac{1}{2\pi c} \int_{0}^{+\infty} dk \sin(kct) J_0(kx), \quad (193) \]

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where we have used the symmetry properties of sine and cosine, and the integral form of the 0th-order Bessel function of the first kind $J_0$ [Abramowitz and Stegun, 1964, eq. (9.1.18), and appendix C here]. The remaining integral in (193) is solved via eq. (11.4.38) of Abramowitz and Stegun [1964], resulting in

$$G_{2D}(x, t) = \frac{1}{2\pi c^2} \frac{H \left( t - \frac{z}{c} \right)}{\sqrt{t^2 - x^2/c^2}}$$  \hspace{1cm} (194)$$

[Sanchez-Sesma and Campillo, 2006; Harmon et al., 2008], where $H$ denotes the Heaviside function. Eq. (194), as well as other time-domain formulae for the Green’s function, is only physically meaningful for $t > 0$. Dimensional analysis of (194) shows that $G_{2D}$ in this formulation has units of time over squared distance. The Fourier transform of $G_{2D}(x, t)$ is inferred from eq. (9.1.24) of Abramowitz and Stegun [1964], after applying our definition (162) to eq. (194):

$$G_{2D}(x, \omega) = \frac{1}{\sqrt{8\pi^2 c^2}} \int_1^\infty dt \frac{e^{-i\omega t}}{\sqrt{t^2 - x^2/c^2}}$$

$$= \frac{1}{\sqrt{8\pi^2 c^2}} \int_1^\infty dt' \frac{e^{-i\omega x/c t'}}{\sqrt{t'^2 - 1}}$$

$$= -\frac{1}{4\sqrt{2\pi}c^2} \left[ Y_0 \left( \frac{\omega x}{c} \right) + iJ_0 \left( \frac{\omega x}{c} \right) \right]$$

$$= \frac{1}{4i\sqrt{2\pi}c^2} H_{(2)}^0 \left( \frac{\omega x}{c} \right)$$  \hspace{1cm} (195)$$

[Sanchez-Sesma and Campillo, 2006; Harmon et al., 2008], where $Y_0$ and $H_{(2)}^0$ denote the 0th-order Bessel function of the second kind and the 0th-order Hankel function of the second kind [Abramowitz and Stegun, 1964, and appendix C here]. Based upon the far-field/high-frequency asymptotic form (175) of $H_{(2)}^0$, we can also write

$$G_{2D}(x, \omega) \approx \frac{1}{4i\pi c^{3/2}} \frac{e^{-i\left( \frac{\omega}{c} - \frac{\pi}{4} \right)}}{\sqrt{\omega x}}$$  \hspace{1cm} (196)$$

which is analogous to eq. (14) of Snieder [2004].

Eq. (9.1.24) of Abramowitz and Stegun [1964] is only valid for positive frequency $\omega > 0$, and so is, as a consequence, our relation (195). We know, however, that $G_{2D}(x, t)$ is a real-valued function: the relationship $G_{2D}(x, \omega) = G_{2D}^*(x, -\omega)$ then holds, and allows us to define $G_{2D}$ in the entire frequency domain.
E.4 Inverse transform of the solution to 3-D space

We now substitute (191) into the 3-D version of (187), and find

$$G_{3D}(x, t) = \frac{1}{8\pi^3} \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \int_{-\infty}^{+\infty} dk_3 \frac{\sin(kct)}{kc} e^{ik \cdot x}. \tag{197}$$

The integral in (197) is simplified by switching from $k_1, k_2, k_3$ to spherical coordinates $k, \xi, \chi$, with the $\xi=0$-direction coinciding with that of $x$. Then $dk_1 dk_2 dk_3 = -k^2 dk \, d\chi \, d(\cos \xi), \quad k \cdot x = kx \cos \xi$, and

$$G_{3D}(x, t) = \frac{1}{2\pi^2 cx} \int_0^{+\infty} dk \sin(kct) \sin(kx). \tag{198}$$

The $k$-integral in (198) is solved via the equality $\sin \alpha \sin \beta = \frac{1}{2} \left\{ \cos(\alpha - \beta) - \cos(\alpha + \beta) \right\}$, which gives

$$G_{3D}(x, t) = \frac{1}{4\pi^2 cx} \int_0^{+\infty} dk \left\{ \cos[k(ct - x)] - \cos[k(ct + x)] \right\}$$

$$= \frac{1}{4\pi^2 cx} \lim_{z \to \infty} \left\{ \frac{\sin[k(ct - x)]}{ct - x} \right\}_{k=0}^{k=z} - \left\{ \frac{\sin[k(ct + x)]}{ct + x} \right\}_{k=0}^{k=z} \tag{199}$$

$$= \frac{1}{4\pi cx} \left[ \delta \left( t - \frac{x}{c} \right) - \delta \left( t + \frac{x}{c} \right) \right],$$

where we have used the property of the Dirac $\delta$ function that $\delta(x) = \frac{1}{\pi} \lim_{z \to \infty} \frac{\sin(xz)}{x}$ [e.g., Weisstein, 1999-2013]. In our formulation both $t$ and $x$ are positive, so that $\delta \left( t + \frac{x}{c} \right) = 0$ and we are left with

$$G_{3D}(x, t) = \frac{1}{4\pi c} \frac{\delta(t - \frac{x}{c})}{x}. \tag{200}$$

[e.g., Aki and Richards, 2002, chapter 4]. Notice that, since the dimension of the Dirac $\delta$ is the inverse of that of its argument, that of $G_{3D}$ is one over squared distance. According to (162), $\mathcal{F}[\delta(t - t_0)] = \frac{1}{\sqrt{2\pi}} e^{-i\omega t_0}$, and consequently

$$G_{3D}(x, \omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{4\pi c} \frac{e^{-i\omega \frac{x}{c}}}{x}. \tag{201}$$
E.5 Green’s problem as inhomogeneous equation

The Green’s problem is also often written as eq. (1) plus an impulsive forcing term, i.e.

\[ \nabla^2 g - \frac{1}{c^2} \frac{\partial^2 g}{\partial t^2} = \delta(x)\delta(t), \]

(202)

with initial conditions

\[ g(x, 0) = 0, \]

(203)

\[ \frac{\partial g}{\partial t}(x, 0) = 0. \]

(204)

The solution \( g \) to this problem is related to the solution \( G \) of (180)-(182); namely

\[ g(x, t) = \int_0^t ds G(x, t - s). \]

(205)

We demonstrate that (205) solves (202)-(204) via Duhamel’s principle [e.g. Hildebrand, 1976; Strauss, 2008]: let us consider the more general case

\[ \nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = h(x, t), \]

(206)

\[ u(x, 0) = 0, \]

(207)

\[ \frac{\partial u}{\partial t}(x, 0) = 0, \]

(208)

with \( h \) an arbitrary forcing term. Suppose that a solution \( v(x, t) \) to the following homogeneous problem, closely related to (206)-(208), can be found:

\[ \nabla^2 v - \frac{1}{c^2} \frac{\partial^2 v}{\partial t^2} = 0, \]

(209)

\[ v(x, 0; s) = 0, \]

(210)

\[ \frac{\partial v}{\partial t}(x, 0; s) = h(x, s). \]

(211)

(In practice, a solution of eq. (209) that satisfies the initial condition (211) must be determined for any possible value of the parameter \( s \) in (211); \( s \) replaces \( t \) in the expression of \( h \) first encountered in (206).) Then, the following relation between \( u \) and \( v \) holds:

\[ u(x, t) = \int_0^t ds v(x, t - s; s). \]

(212)
One can verify that (212) solves (209)-(211) by direct substitution, applying Leibniz’s rule for differentiating under the integral sign. Equation (205) is a particular case of (212).

Result (212) can also be combined with (186) to write the solution of the general inhomogeneous problem (206) in terms of the Green’s function $G$ (e.g., $G_{2D}$ or $G_{3D}$). Replacing $v$ in eq. (212) with expression (186),

$$u(x, t) = \int_0^t ds \int_{\mathbb{R}_d} d^d x' G(x, x', t - s; s) h(x', s) \quad (d = 2, 3). \quad (213)$$