In this course we illustrate various approaches to the problem of predicting the displacement of the Earth (calculating theoretical seismograms) in response to a given seismic excitation (earthquake): the forward problem of global seismology. As the Earth is bounded by an outer surface approximately free of stresses, one approach to solve the said forward problem is to treat it as a boundary-value problem like that of the guitar string; the stress-free surface of the Earth then plays the same role as the fixed ends of the string in the previous lecture, and the requirement that the standing-wave solutions that we likewise find satisfy the no-stress condition at the outer surface results in the identification of a discrete set of suitable solutions, with associated eigenfrequencies: the normal modes (free oscillations) of the Earth.

Self-Gravitation and Linearization of the Momentum Equation

Gravity

So far we have written the Earth’s equation of motion (momentum conservation) in a simplified, approximate form, neglecting the effects of gravity and the perturbations to the gravity field caused by the Earth’s deformation $u(r, t)$ itself (“self-gravitation”). Those terms, however, turn out to be significant, and should not be neglected in a realistic simulation.

The acceleration felt at any point in space $r$ due to the gravitational attraction of a distribution of mass $\rho(r')$ is given by Newton’s law of gravitation,

$$\ddot{u}(r, t) = \int_V \frac{r' - r}{|r' - r|^3} G \rho(r') d^3r', \quad (1)$$

with $G$ denoting Newton’s constant. Equation (1) is then simplified introducing a gravitational potential $\Phi(r)$ such that

$$\nabla \Phi(r, t) = -\ddot{u}(r, t). \quad (2)$$

Then, taking the divergence of (1),

$$\nabla^2 \Phi(r, t) = -\int_V \nabla \cdot \left[ \frac{r' - r}{|r' - r|^3} \right] G \rho(r') d^3r'. \quad (3)$$
Note that, for \( r' \neq r \), we can write
\[
\nabla \cdot \left[ \frac{r' - r}{|r' - r|^3} \right] =
\]
\[
= \left( \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'} \right) \cdot \left[ \frac{(x' - x)(y' - y)(z' - z)}{|r' - r|^5} \right]
\]
\[
= \frac{\partial}{\partial x} \left( \frac{x' - x}{|r' - r|^3} \right) + \frac{\partial}{\partial y} \left( \frac{y' - y}{|r' - r|^3} \right) + \frac{\partial}{\partial z} \left( \frac{z' - z}{|r' - r|^3} \right)
\]
\[
= - \frac{3}{|r' - r|^5} + (x' - x) \frac{\partial}{\partial x} \left\{ \frac{1}{(x' - x)^2 + (y' - y)^2 + (z' - z)^2} \right\} + \text{etc}.
\]
which equals 0 everywhere except at \( r' = r \). This means that the integral at the right hand side of (3) can be calculated over an infinitely small, spherical volume centered on \( r' = r \); the divergence theorem (earlier lecture) is applied to this integral, noting that the unit-vector perpendicular to the surface of the sphere \( \hat{n} = \frac{r' - r}{|r' - r|} \), and the surface element on the spherical surface \( d^2r' = |r' - r|^2 \sin(\theta) d\theta d\phi \), with \( \theta \) and \( \phi \) spherical coordinates. \( \rho(r') \) is approximately constant and equal to \( \rho(r) \) within the sphere in question. Poisson’s equation follows,
\[
\nabla^2 \Phi = 4\pi G \rho.
\]
(5)

We account for gravity including in the usual equation of motion a body-force-density term equal to \( \rho \) times the acceleration (2), i.e.
\[
\rho \frac{\partial^2}{\partial t^2} \mathbf{u}(r,t) = \nabla \cdot \mathbf{\tau}(r,t) - \rho \nabla \Phi(r,t).
\]
(6)

Notice that in both (5) and (6) \( \rho \) changes in time as a result of the deformations \( \mathbf{u}(r,t) \).
(Compare (6) with (2.93) or (2.117) of Dahlen & Tromp, keeping in mind that Coriolis and centrifugal forces have a smaller effect and can be neglected.)

**Linearization**

In normal-mode literature it is then customary to simplify equations (6) and (5) by linearization, that is to say considering that the quantities \( \rho, \Phi, \mathbf{\tau} \) can reasonably be written
\[
\rho = \rho_0 + \rho'
\]
(7)
\[
\Phi = \Phi_0 + \Phi'
\]
(8)
\[
\mathbf{\tau} = \mathbf{\tau}_0 + \mathbf{\tau}'
\]
(9)

where \( \rho', \Phi', \mathbf{\tau}' \) are small perturbations with respect to the initial values \( \rho_0, \Phi_0, \mathbf{\tau}_0 \). After substituting (7) through (9) into (6) and (5), neglecting second-order quantities and making use of the static equilibrium condition \( \nabla \cdot \mathbf{\tau}(r,t) = \rho \nabla \Phi(r,t) \), one finds the linearized momentum equation
\[
\rho_0 \frac{\partial^2}{\partial t^2} \mathbf{u}(r,t) = \nabla \cdot \mathbf{\tau}'(r,t) - \rho_0 \nabla \Phi'(r,t) - \rho' \nabla \Phi_0(r,t)
\]
(10)
(compare with eq. (3.56) or (3.60) of Dahlen & Tromp), and linearized Poisson’s equation
\[
\nabla^2 \Phi' = 4\pi G \rho'
\]
(11)

(sometimes further simplified with the incompressibility assumption \( \rho' = 0 \)).
Spherically Symmetric, Non-Rotating, Elastic and Isotropic Earth

Models of this type provide a first approximation, relatively realistic and relatively simple, to the free oscillations of the true (slightly aspherical, slightly anisotropic, slightly anelastic) Earth. The assumptions of elasticity and isotropy are implicit in our choice of the constitutive relation between stress and deformation (see previous lectures); the neglect of lateral heterogeneities comes into play when horizontal gradients of $\rho, \lambda, \mu$ and the like are neglected, to find a simple, explicit form for the momentum and Poisson’s equations (i.e., eqs. (20) through (23) below).

In analogy with most textbooks, let us Fourier-transform the Earth’s linearized equation of motion (10) to the frequency domain,

$$-\rho \partial_\omega^2 \mathbf{u}(r, \omega) = \nabla \cdot \mathbf{\tau}'(r, \omega) - \rho_0 \nabla \Phi'(r, \omega) - \rho \nabla \Phi_0(r, \omega)$$  \hspace{1cm} (12)

where $\omega$ denotes frequency and the superscript $\sim$ (omitted in the following) indicates the Fourier-transform of a function. Poisson’s equation (11) holds the same form in the frequency domain.

Fourier-transforming is in a sense equivalent to separating the time-dependence from the space-dependence of the solution, like we did in the case of the guitar string; a standing-wave description of the solution follows naturally.

In practice, the Ansatz

$$\mathbf{u}(r, \omega) = U_{lm}(r, \omega) \mathbf{P}_{lm}(\theta, \phi) + V_{lm}(r, \omega) \mathbf{B}_{lm}(\theta, \phi) + W_{lm}(r, \omega) \mathbf{C}_{lm}(\theta, \phi)$$  \hspace{1cm} (13)

(no summation over $l, m$) is substituted into (12), and

$$\Phi(r, \omega) = P_{lm}(r, \omega) Y_{lm}(\theta, \phi)$$  \hspace{1cm} (14)

is substituted into the frequency-domain version of (11) (Dahlen & Tromp, section 8.6.1).

For practical reasons we now employ a system of spherical-polar coordinates, $r = r, \theta, \phi$, with its origin at the center of the Earth. Vector spherical harmonics are defined

$$\mathbf{P}_{lm}(\theta, \phi) = \hat{r} Y_{lm}(\theta, \phi),$$  \hspace{1cm} (15)

$$\mathbf{B}_{lm}(\theta, \phi) = \frac{1}{\sqrt{l(l+1)}} \nabla_1 Y_{lm}(\theta, \phi),$$  \hspace{1cm} (16)

$$\mathbf{C}_{lm}(\theta, \phi) = -\frac{1}{\sqrt{l(l+1)}} (\hat{r} \times \nabla_1) Y_{lm}(\theta, \phi),$$  \hspace{1cm} (17)

with $Y_{lm}(\theta, \phi)$ scalar spherical harmonics, and $\hat{r}$ the unit vector perpendicular to the unit sphere, directed outwards. The surface gradient $\nabla_1$ can be defined directly,

$$\nabla_1 = \hat{\theta} \partial_\theta + \hat{\phi} \frac{1}{\sin(\theta)} \partial_\phi$$  \hspace{1cm} (18)

or by specifying its relation to the gradient $\nabla$,

$$\nabla = \hat{r} \partial_r + \frac{1}{r} \nabla_1.$$  \hspace{1cm} (19)

The Ansatz (13), (14) is chosen so that the problem of finding two functions of three variables, $\mathbf{u}(r, \theta, \phi)$ and $\Phi(r, \theta, \phi)$, satisfying (12) and (11), is replaced by that of integrating,
for each couple of values of \( l \) and \( m \), a system of four second-order differential equations for \( P_{lm}(r), U_{lm}(r), V_{lm}(r), \) and \( W_{lm}(r) \) (not depending on \( \theta \) or \( \phi \)). The latter endeavour, in fact, turns out to be much easier.

Substituting (13) into (12) involves writing \( \tau' \) in terms of \( \mathbf{u} \): the assumption is usually made that the initial stress \( \tau_0 \) be purely hydrostatic (\( \tau_0 = -p_0 \mathbf{I} \) with \( p_0 \) hydrostatic pressure), and the constitutive relation (see previous lectures) is accordingly corrected. Likewise, based on physical considerations, perturbations \( \rho' \) in density can be written in terms of unperturbed density \( \rho_0 \) and displacement \( \mathbf{u}, \rho' = -\nabla \cdot (\rho_0 \mathbf{u}) \). Then, making use of the orthogonality properties of vector spherical harmonics, (12) and (11) can be written

\[
\frac{1}{r^2} \frac{d}{dr} \left[ r^2 (\lambda + 2 \mu) \dot{U} + \lambda r (2 U - \sqrt{l(l+1)} V) \right] + \frac{1}{r} \left[ (\lambda + 2 \mu) \dot{U} + \frac{\lambda}{r} \left( 2 U - \sqrt{l(l+1)} V \right) \right] \\
- 3 \left( \frac{\lambda + 2 \mu}{3} \right) \frac{1}{r} \left( \dot{U} + \frac{2}{r} U - \sqrt{l(l+1)} V \right) - \sqrt{l(l+1)} \frac{\mu}{r} \left( \dot{V} - \frac{1}{r} V + \sqrt{l(l+1)} U \right) \\
+ \omega^2 \rho U - \rho \left[ \dot{P} + (4 \pi \rho' - \frac{4g}{r}) U + \frac{\sqrt{l(l+1)} g}{r} V \right] = 0, \tag{20}
\]

\[
\frac{1}{r^2} \frac{d}{dr} \left[ \mu r^2 \left( \dot{V} - \frac{1}{r} V - \sqrt{l(l+1)} U \right) \right] + \frac{\mu}{r} \left( \dot{V} - \frac{1}{r} V - \sqrt{l(l+1)} U \right) \\
+ \frac{\sqrt{l(l+1)} \lambda}{r} \dot{U} + \frac{\sqrt{l(l+1)} \lambda}{r^2} \left( 2 U - \sqrt{l(l+1)} V \right) \\
- \left[ \omega^2 \rho - (l(l+1) - 2) \frac{\mu}{r^2} \right] V - \frac{\sqrt{l(l+1)} \rho}{r} (P + gU) = 0, \tag{21}
\]

\[
\frac{1}{r^2} \frac{d}{dr} \left[ \mu r^2 \left( \dot{W} - \frac{W}{r} \right) \right] + \frac{\mu}{r} \left( \dot{W} - \frac{W}{r} \right) + \left[ \omega^2 \rho - \left( \sqrt{l(l+1)} - 2 \right) \frac{\mu}{r^2} \right] W = 0, \tag{22}
\]

\[
\dot{P} + \frac{2}{r} \dot{P} - \frac{l(l+1)}{r^2} P = -4 \pi G \rho U - 4 \pi G \rho \left( \dot{U} + \frac{2 U - \sqrt{l(l+1)} V}{r} \right), \tag{23}
\]

where the superscript \( \cdot \) now denotes derivation with respect to \( r, \ g = \frac{\partial}{\partial r} \Phi, \rho = \rho_0, \) and the subscript \( \ \text{lm} \) was omitted from \( U_{lm} \) etc. for brevity. Compare with equations (8.43) through (8.45), and (8.53) from Dahlen & Tromp.

From these cumbersome equations two important results are apparent. They are referred to as toroidal-spheroidal decoupling and degeneracy.

**Toroidal-Spheroidal Decoupling**

While to find \( U_{lm}, V_{lm} \) and \( P_{lm} \) we must integrate simultaneously (20), (21) and (23), equation (22) can be solved independently for \( W_{lm} \), which does not appear in the other radial equations: we say that (22) is decoupled from (20), (21) and (23). Inspection of equations (13) and (17) shows that displacements associated with \( W_{lm} \) are parallel to the surface of the Earth, or toroidal; displacements associated with \( U_{lm} \) and \( V_{lm} \) are called, instead, spheroidal; hence the expression toroidal-spheroidal decoupling \(^1\).

\(^1\)Like the equations of motion, boundary conditions at the Earth’s surface decouple into two equations involving \( U, V, \) and one equation involving \( W \); see Dahlen & Tromp, equations (8.46) through (8.52).
Figure 1: Radial eigenfunctions $W_{0l}$ of the first ten “fundamental” ($n = 0$) toroidal modes in PREM. Vertical axis is depth beneath the surface of the Earth; the location of the “670” km discontinuity and core mantle boundary (“CMB”) are indicated. The acronym $nT_l$ identifies the mode, “T” standing for “toroidal”. Figures included in this document are taken from chapter 8 of Dahlen and Tromp.

In a realistic, spherically symmetric Earth model the radial equations (20) through (22) are typically solved numerically. As boundary conditions (regularity of the solution at the center of the Earth; no stresses acting on the outer surface) are imposed, for each couple $l, m$ a discrete set of acceptable values of $\omega$ (eigenfrequencies) is determined.

Spheroidal and toroidal equations are solved separately and independently, and give rise to two independent sets of eigenfrequencies, dubbed spheroidal and toroidal eigenfrequencies, and denoted $\omega_{nlm}^S$ and $\omega_{nlm}^T$, respectively. Remark that we have introduced a third index $n$, to distinguish the infinite possible eigenfrequencies associated with a given couple $l, m$.

Each spheroidal eigenfrequency $\omega_{nlm}^S$ is associated with a couple of spheroidal eigenfunctions $U_{nlm}(r), V_{nlm}(r)$, solutions of (20), (21) with $\omega = \omega_{nlm}^S$.

Likewise, each toroidal eigenfrequency $\omega_{nlm}^T$ is associated with one toroidal eigenfunction $W_{nlm}(r)$, solution of (22) with $\omega = \omega_{nlm}^T$.

The eigenfunctions $U_{nlm}(r), V_{nlm}(r)$ with eigenfrequencies $\omega_{nlm}^S$, or $W_{nlm}(r)$, with eigenfrequency $\omega_{nlm}^T$, correspond in the elastic-string example to the eigenfunctions $\xi_k(x)$, with eigenfrequencies $k\pi c/L$ (notation defined in previous lecture).

Modes associated with $n = 0$ are called fundamental modes. Modes associated with $n > 0$ are called overtones. Like in the elastic-string case, the number of “nodes” of a mode coincides with its overtone number $n$ (see, for example, the toroidal eigenfunctions in figure 1).

**Degeneracy and Splitting**

Unlike the harmonic degree $l$, the harmonic order $m$ does not appear anywhere in eqs. (20) through (23). This means that the eigenfunctions $U_{lm}^n, V_{lm}^n, W_{lm}^n$ and $P_{lm}^n$, and eigenvalues $\omega_{nlm}^T$ and $\omega_{nlm}^S$ are constant with respect to $m$. The set of $2l + 1$ modes associated with a given value of $l$ (and all integer values of $m$ from $-l$ to $+l$) form a multiplet.

Degeneracy is limited to spherically symmetric (laterally homogeneous) Earth models, which are only realistic in the first approximation. When the normal modes of more realistic, slightly aspherical (laterally heterogeneous) Earth models are computed, modes and eigenfrequencies become $m$-dependent: an effect known as splitting (see for example the split
Figure 2: Examples of the vibrations associated with different spheroidal modes. From http://web.ics.purdue.edu/~nowack/eas557.html

Figure 3: Examples of the vibrations associated with different toroidal modes. From http://web.ics.purdue.edu/~nowack/eas557.html
eigenfrequency of mode $1S_4$ in figure 4).

**Green’s Tensor and Synthetic Seismograms**

It can be shown that the eigenfunctions $U_{lm}^n(r)P_{lm}(\theta, \phi) + V_{lm}^n(r)B_{lm}(\theta, \phi)$ and $W_{lm}^n(r)C_{lm}(\theta, \phi)$ form a complete basis of solutions to the boundary problem associated with (12) plus the no-stress boundary conditions at the outer surface, i.e. any displacement field of an Earth with stress-free outer surface can be written as their linear combination. In the following, for simplicity and in analogy with Dahlen & Tromp, we shall collapse the three indexes $n, l, m$ to one index $k$, also used to distinguish spheroidal and toroidal modes, so that we can write all eigenfunctions indifferently as $s_k(r)$.

It can also be shown that $s_k(r)$ are orthonormal, or

$$\int_V \rho_0 s_k(r) \cdot s_{k'}(r) dV = \delta_{kk'}.$$  \hfill (24)

We now want to use normal-mode formalism to build a mathematical tool that, given the mechanism of an earthquake (geometry of the fault and slip on the fault), will provide us with the displacement field anywhere in the Earth, at any time following the earthquake. As we have learned in a previous lecture, thanks to Betti’s theorem this can be done by solving the Green’s problem associated with the Earth, i.e. finding the response of the Earth to an impulsive excitation. Previously, we have represented the impulsive excitation with a force density included in the momentum equation; equivalently, it can be represented as an initial condition on the displacement. Said $G(r, r'; t)$ the Green’s tensor (again in the time domain), solving the Green’s problem consists of solving eq. (10) after replacing $u$ with $G$, and with initial conditions corresponding to an impulsive source at $t = 0$,

$$G(r, r'; 0) = 0,$$  \hfill (25)

$$\frac{\partial}{\partial t} G(r, r'; 0) = \frac{1}{\rho_0} I \delta(r - r').$$  \hfill (26)

Because the modes form a complete basis of eigensolutions, $G$ can be written as a linear combination of modes,

$$G(r, r'; t) = \sum_k s_k(r) \left[ a_k(r') \cos(\omega_k t) + b_k(r') \sin(\omega_k t) \right].$$  \hfill (27)

Substituting (27) into (25) and (26) we find, respectively,

$$\sum_k s_k(r) a_k = 0,$$  \hfill (28)

$$\sum_k \omega_k s_k(r) b_k = \frac{1}{\rho_0} I \delta(r - r').$$  \hfill (29)

After multiplying both sides by $s_k(r)$, integrating over the Earth’s volume $V$, and making use of the orthonormality relation (24), (28) and (29) become

$$a_k = 0$$  \hfill (30)

for all values of $k$, and

$$b_k = \frac{1}{\omega_k} s_k(r').$$  \hfill (31)
The Green’s tensor of a non-rotating Earth is therefore given in terms of the normal-mode eigenfrequencies and eigenfunctions by

\[ G(r, r'; t) = \sum_k \frac{1}{\omega_k} s_k(r)s_k(r') \sin(\omega_k t). \]  

(32)

After \( G \) has been so determined, the response of the Earth model in question to any seismic excitation can be modeled via Betti’s theorem (numerically in general, analytically in some simplified cases).

**Lateral Heterogeneities and Splitting Functions.**

The displacement of a slightly laterally heterogeneous (“perturbed”) Earth can be written as a linear combination of the modes of a spherically symmetric (“unperturbed”) Earth. For this result to be general, all unperturbed-Earth modes have to be considered.

One can assume, however, that the contribution of one multiplet to the displacement in the perturbed Earth can be written as the linear combination of unperturbed-Earth modes within that multiplet only (if this is true, the multiplet is said to be “isolated”).

In this assumption, the contribution of an isolated multiplet to the displacement in the perturbed Earth is written

\[ u(t) = \text{Re} \left[ r^T \cdot \exp(iHt) \cdot s \exp(i\omega t) \right], \]

(33)

where \( \text{Re} \) indicates that only the real part of the expression at the right hand side is considered; \( \omega \) is the degenerate eigenfrequency of the multiplet in the unperturbed Earth; the vectors \( s \) and \( r \), dubbed source vector and receiver vector, respectively, have \( 2l+1 \) entries each; their \( m \)-th entries are defined \( M : \nabla u_m \) (source) and \( \nu \cdot u_m \) (receiver). \( \nu \) is the polarization vector (notice eq. (33) is scalar: we consider only one component of the seismogram at a time) and the tensor \( M \) represents the effect of the magnitude and geometry of the seismic source.

Naturally, most quantities in equation (33) depend on the indexes \( n, l \); I have omitted them because only one multiplet (i.e. one value of \( n \), one value of \( l \)) is considered. For example, \( u_m = u_{nlm} \); for a toroidal mode, \( u_m = W_{im}(r)C_{lm}(\theta, \phi) \).

The \((2l+1) \times (2l+1)\) matrix \( \exp(iHt) \) is called “splitting matrix”, and governs multiplet splitting. In the interest of simplicity I have written it in an implicit form; I need now to specify

\[ H_{ij} = \omega \sum_{s,t} c_{st} \int_{\Omega} Y_{li}^* Y_{st} Y_{lj} d\Omega, \]

(34)

where \( \Omega \) denotes the unit sphere, and the splitting coefficients

\[ c_{st} = \int_0^a (K_{p}^{st} \delta v_p^{st} + K_{s}^{st} \delta v_s^{st} + K_{p}^{st} \delta \rho^{st}) dr. \]

(35)

\( \delta v_p^{st} \), \( \delta v_s^{st} \) and \( \delta \rho^{st} \) are the \( st \) harmonic coefficients of perturbations in the Earth’s compressional and shear velocity, and density, respectively; \( K_{p}^{st} \), \( K_{s}^{st} \) and \( K_{p}^{st} \) are the corresponding partial derivatives, or sensitivity kernels (Seismic Tomography course).

---

2 Eq. (1) of He and Tromp 1996, “Normal-mode constraints on the structure of the Earth”, after neglecting the Earth’s anelastic damping.

3 Eq. (2) of He and Tromp 1996, after neglecting ellipticity and rotational effects.

4 Eq. (6) of He and Tromp 1996, after neglecting perturbations in the Earth’s internal discontinuities.
Replacing (35) into (34) and the resulting expression for H into (33), a relation is established between the measured displacement u and perturbations in the Earth’s structure, and, in the presence of sufficient data, an inverse problem can be solved to derive the latter from the former (Seismic Tomography course).

The inverse problem can also be subdivided into two subsequent steps; after substituting (34) into (33), I can invert (33) for the splitting coefficients c_{st}. The resulting values can then be substituted into (35), and (35) in this new form can be inverted for \( \delta v_{pl}^{st} \), etc. This is feasible, of course, so long as a number of different measurements (i.e. different sources and/or different receivers) are available for the considered multiplet (the problem must be overdetermined).

In the literature, you shall often find the splitting function\(^5\)

\[
\sum_{s=0,2,4,...}^{2l} \sum_{t=-s}^{s} c_{st} Y_{st}
\]

(36)
as a convenient way to visualize normal-mode splitting.

Notice that the sum in equation (36) is limited to even values of s. Coefficients c_{st} with s odd remain undefined here, because the integral in equation (34) can be proved to be 0 when s is odd (and when \( t \neq i - j \)). This is the main limit of the isolated-multiplet approximation: in tomographic applications, only the even harmonic order of the Earth’s lateral structure can be constrained.

Further Reading


\(^5\)First introduced by Woodhouse, Giardini & Li, 1986.
Figure 4: The spectrum of the Earth’s long-period radial displacement. Measured at station TUC (Tucson, Arizona), after the June 9, 1994 Bolivia earthquake. Note the visible split of mode $1S_4$. From *Dahlen & Tromp*. 