Global surface-wave tomography

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Love and Rayleigh waves, radial anisotropy

Whenever an elastic medium is bounded by a free surface, coherent waves arise that travel along that surface; the amplitude of those surface waves decays with increasing distance from the surface (that is to say, in the Earth, with increasing depth). Two distinct types of surface waves are observed: Love waves, whose associated displacement is parallel to the free surface and perpendicular to the direction of propagation (toroidal), and Rayleigh waves, with displacement on a plane perpendicular to Love-wave displacement (spheroidal). The speed with which surface waves propagate is related to, but does not coincide with, compressional and/or shear velocities in the region of the medium close to the free surface (in the Earth, the upper mantle). In a transversely isotropic (i.e. radially anisotropic) Earth, the velocity of seismic shear waves is defined by the two parameters $v_{SH}$, which coincides with the speed of a horizontally polarized, horizontally traveling shear wave, and $v_{SV}$, which coincides with the speed of a vertically polarized, horizontally traveling shear wave. Because of the toroidal nature of Love waves, Love wavespeed depends primarily on horizontally polarized shear velocity; Rayleigh wavespeed likewise depends on compressional and vertically polarized shear velocities.

It is important to make the distinction between $v_{SH}$ and $v_{SV}$ early in our formulation, because the Earth, at least within part of the upper mantle, is strongly radially anisotropic: theories of surface-wave propagation that do not account for this effect will not work. Other, more complicated forms of anisotropy can be left for later discussion.

The range of depths over which a surface wave has nonzero amplitude is a function of the frequency of the wave itself. Lower-frequency surface waves typically “sample” deeper than higher-frequency ones (and, at any given frequency, Rayleigh waves sample deeper than Love). Depending on the elastic properties of rocks in the sampled depth-range, surface waves of different frequencies will propagate at different speeds: as a result, multi-frequency surface-wavepackets are clearly observed to disperse as they propagate away from their source. From the point of view of tomography, this means that surface-wave signal filtered at different frequencies will correspondingly provide information about different regions (depth ranges) of the Earth.

Surface-wave ray theory

Surface waves can also be thought of as elastic waves propagating on an empty spherical shell, their speed of propagation a function of latitude and longitude only. In this idealized
monochromatic scenario, the three-dimensional problem of seismic wave propagation is reduced to two dimensions, and we apply (as with body waves) the infinite-frequency, optics ray-theory approximation, to write

$$\delta t(\omega) = - \int_{\text{ray path}} \frac{\delta c(\theta(s), \phi(s); \omega)}{c^2(\theta(s), \phi(s); \omega)} ds,$$

where $\delta t$ denotes travel-time anomaly, $\omega$ frequency, $c$ is surface-wave velocity at a given frequency (i.e., phase velocity), and $\theta$ and $\phi$ are colatitude and longitude, respectively; $\theta = \theta(s)$ and $\phi = \phi(s)$ are equations describing the ray path, determined via the laws of optics, along which the integration at the right hand side is to be performed, with $s$ incremental length along the ray path. $\delta$ before a symbol indicates a perturbation to the associated quantity. In the following we shall see how equation (1) can be used to derive, from surface wave observations, useful information about the properties of the Earth$^1$. Equation (1) holds indifferently for Love and Rayleigh waves, and so does the formulation that follows.

$^1$This outline of surface wave theory is deliberately simplistic. More detailed accounts are given in most global seismology textbooks; you can look for example at Lay and Wallace, Modern Global Seismology; Aki and Richards, Quantitative Seismology; Dahlen and Tromp, Theoretical Global Seismology. A classic reference to surface wave theory is the long article by Takeuchi and Saito, Seismic surface waves, in Methods in Computational Physics 11, ed. by B. A. Bolt, Academic Press 1972. For those interested in the relationship between surface waves and radial anisotropy, a good starting point is Dziewonski and Anderson, Preliminary reference Earth model, published on Phys. Earth Planet. Int., vol. 25, pages 297–336 (1981).
Phase-velocity maps (a 2D inverse problem)

The speed at which a surface wave propagates depends on its frequency. Lower frequency waves are faster: they propagate over a wider depth range, hence sample deeper material, and in the Earth seismic velocities generally increase with increasing depth. The fact that surface waves are dispersive implies that a different phase velocity, function of $\theta$ and $\phi$, is defined, for Love and Rayleigh waves independently, at each considered frequency $\omega$. At a chosen frequency, equation (1) can be employed to formulate an inverse problem.

Phase velocity heterogeneity $\delta c$ is parameterized,

$$\delta c(\theta, \phi) = \sum_{i=1}^{N} x_i f_i(\theta, \phi),$$  \hspace{1cm} (2)

where $f_i$ denotes the $i$-th basis function from a set of $N$, now only dependent on $\theta$ and $\phi$, and the coefficients $x_i$ are unknown. If (2) is substituted into (1),

$$\delta t = -\sum_{i=1}^{N} x_i \int_{\text{ray path}} \frac{f_i(\theta, \phi)}{c^2(\theta, \phi)} \, ds,$$  \hspace{1cm} (3)

where the integral can be calculated. (For simplicity, I have omitted to specify the dependence of $\theta, \phi$ on $s$ according to the ray-path equation.) If the chosen reference model is, like PREM, spherically symmetric, Love and Rayleigh wave phase velocities are constant at any given frequency, and (3) takes the simpler form

$$\delta t = -\frac{1}{c^2} \sum_{i=1}^{N} x_i \int_{\text{ray path}} f_i(\theta, \phi) \, ds \quad (j = 1, \ldots, M)$$  \hspace{1cm} (4)

(in the body-wave case we encounter a similar equation; integration here is much simpler, $f_i$ being a function of $\theta, \phi$ only, not of depth).

If a database of $M \gg 1$ observations of surface wave phase anomaly is given, said $\delta t_j$ the $j$-th datum and

$$A_{ji} = -\frac{1}{c^2} \int_{\text{ray path}_j} f_i(\theta, \phi) \, ds,$$  \hspace{1cm} (5)

we are left with the familiar linear inverse problem

$$A \cdot x = \delta t,$$  \hspace{1cm} (6)

which we can solve in the same way as in the body-wave case.

Your homework assignment will consist of running a set of FORTRAN routines, which will compute $A$ (equation (5)) and $A^T \cdot A$ for a given set of surface-wave $\delta t$ anomalies, and then solve the inverse problem by either one of two inversion algorithms (LSQR, Cholesky factorization).

How to measure a surface wave phase anomaly

The travel time of a $P$ or $S$ wave can be “picked” by visual inspection of the seismogram, because direct $P$ and $S$ generally have larger amplitude than other phases that hit the source at about the same time.
Figure 2: Love wave phase velocity map from phase anomaly measurements at various frequencies (from Ekström, Tromp and Larson, Measurements and global models of surface-wave propagation, *J. geophys. Res.*, vol. 102, pages 8137–8157, 1997).

Figure 3: Rayleigh wave phase velocity map from phase anomaly measurements at various frequencies (from Ekström, Tromp and Larson, 1997).
Measuring surface wave phase anomalies $\delta t(\omega)$ is not so easy: surface waves of different frequencies are all superimposed in teleseismic seismograms, and generally have comparable amplitudes.

$\delta t(\omega)$ is measured from a seismogram assuming phase velocity $c = c(\omega)$ to not depend on location, writing the ray-theory solution for the seismogram as a function of $c(\omega)$, and looking for the function $c(\omega)$ for which the difference between ray-theory solution and observed seismogram is minimum. You can see this as a nonlinear inverse problem, requiring a nonlinear minimization algorithm\(^2\) rather than the least-squares solvers we have seen so far.

Shorter periods (higher frequencies) are more strongly affected by shallow structure, which in the Earth is more heterogeneous (crust): observed higher-frequency surface wave phases will therefore tend to be further away from our reference model predictions than lower-frequency ones. As illustrated in figure 4, it is then convenient to first correct $c(\omega)$ so that the low-frequency component of the seismogram is fit, and then move on to higher frequencies, “anchoring” the high-frequency component to the low-frequency one (requirement that the function $c = c(\omega)$ be smooth).

From the phase velocity $c(\omega)$ one can derive the travel time $t$ at each frequency $\omega$. The difference between this value, and the phase predicted by a reference model, is a phase anomaly measurement $\delta t(\omega)$. The function $c = c(\omega)$ is referred to as “dispersion curve”.

Radially anisotropic models of the upper mantle (a 3-D inverse problem)

The relationship between phase velocity $c$ and Earth structure can be determined in various ways: the conceptually simplest one is to just perturb Earth structure at a given location and depth, forward-model the Earth’s seismic response, and iterate over all possible locations and depths of the perturbation. In practice, so long as the Earth is considered close to spherically symmetric, it is possible to find analytical formulae for this relationship, e.g. via normal-mode theory. In the first approximation, it turns out this relationship is linear, and limited to the structure directly underlying the place at the surface where $c$ is measured. Namely,

$$
\delta c(\omega, \theta, \phi) = \int_0^a \left[ K_{v_{SH}}(\omega, \mathbf{r}) \delta v_{SH}(\mathbf{r}) + K_{v_{SV}}(\omega, \mathbf{r}) \delta v_{SV}(\mathbf{r}) + K_{\nu_{PH}}(\omega, \mathbf{r}) \delta \nu_{PH}(\mathbf{r}) + K_{\nu_{PV}}(\omega, \mathbf{r}) \delta \nu_{PV}(\mathbf{r}) + K_{\rho}(\omega, \mathbf{r}) \delta \rho(\mathbf{r}) \right] d\mathbf{r},
$$

\(7\)

where $\nu_{PV}$ and $\nu_{PH}$, are vertical and horizontal compressional-wave velocities, respectively, $\rho$ is density, and the function $K_x$ is the “sensitivity kernel” associated to the observable $x$. I omit here the complicated analytical expressions for $K_x$. Examples of $K_x$ are given in figures 5 and 6. Equation (7) neglects attenuation and the anisotropy parameter $\eta$ (see, e.g., Dziewonski and Anderson, Preliminary reference Earth model, 1981). For Love-wave fundamental modes\(^3\), $K_{v_{SH}}$ is larger than other sensitivity functions, which could be neglected, i.e.

$$
\delta c_L(\omega, \theta, \phi) = \int_0^a K_{v_{SH}}(\omega, \mathbf{r}) \delta v_{SH}(\mathbf{r}) d\mathbf{r},
$$

\(8\)

\(^2\)Chapter 10 of Numerical Recipes by Press et al.

\(^3\)The concept of fundamental mode vs. overtone is illustrated in the spherical Earth seismology course. In this course we will only talk about fundamental modes.
Figure 4: Fitting a teleseismic surface-wave measurement with a dispersion curve accurate over an increasingly broad range of periods (from Ekström, Tromp and Larson, 1997). (Left) Comparison between the (filtered) observed seismogram (thick line) and the one modeled on the basis of the dispersion measurement (thin line). (Right) The reference (PREM) dispersion curve, and the measured one corresponding to panel e at the left.
with the subscript \( L \) reminding us that this is only valid for Love waves. For Rayleigh-wave fundamental modes, the more important sensitivities are to \( v_{SV}, v_{PV} \) and \( v_{PH} \).

We next see how equation (7) can be used to formulate an inverse problem relating surface-wave observations to unknown, 3-D upper mantle structure. Let us limit ourselves to the theoretically simpler Love-wave case, i.e. eq. (8). The expression (8) for \( \delta c_L(\omega, \theta, \phi) \) can be substituted into equation (1), which then takes the form

\[
\delta t_L(\omega) = - \int_{\text{ray path}} \frac{1}{c_L^2(\theta, \phi; \omega)} \int_0^a K_{v_{SH}}(\omega, r) \delta v_{SH}(r) \, dr \, ds. \tag{9}
\]

If we write \( \delta v_{SH} \) as a linear combination of \( N \) known basis functions \( f_i(r) \) (now also dependent on radius),

\[
\delta v_{SH}(r) = \sum_{i=1}^{N} x_i f_i(r), \tag{10}
\]

and substitute into (9), we find

\[
\delta t_L(\omega) = - \sum_{i=1}^{N} x_i \int_{\text{ray path}} \frac{1}{c_L^2(\theta, \phi; \omega)} \int_0^a K_{v_{SH}}(\omega, r) f_i(r) \, dr \, ds. \tag{11}
\]

Note that if the reference model is chosen to be spherically symmetric, equation (11) takes the simpler form

\[
\delta t_L(\omega) = - \frac{1}{c_L^2(\omega)} \sum_{i=1}^{N} x_i \int_0^a K_{v_{SH}}(\omega, r) \int_{\text{ray path}} f_i(r) \, ds \, dr. \tag{12}
\]

The integral at the right-hand side of eq. (12) can be computed (if, as is usually the case, each \( f_i(r) \) is defined as the product of a function of \( r \) times a function of \( (\theta, \phi) \), the two integrals in (12) can be computed independently). Applying eq. (12) to our database of \( M \gg 1 \) observations of \( \delta t_L \),

\[
\delta t_{Lj}(\omega) = - \frac{1}{c_L^2(\omega)} \sum_{i=1}^{N} x_i \int_0^a K_{v_{SH}}(\omega, r) \int_{\text{ray path}_j} f_i(r) \, ds \, dr \quad (j = 1, \ldots, M). \tag{13}
\]

Said \( B_{ji} = \int_0^a K_{v_{SH}}(\omega, r) \int_{\text{ray path}_j} f_i(r) \, ds \, dr \), we are left with

\[
\mathbf{B} \cdot \mathbf{x} = \delta t_L, \tag{14}
\]

analogous to equation (6). Least-squares solving (14) to determine \( \mathbf{x} \) will yield a 3-D image of horizontally polarized shear velocity \( v_{SH} \) in the Earth’s upper mantle. The same treatment can be applied to Rayleigh waves and \( v_{SV} \) (and \( v_{PH}, v_{PV} \)), resulting in radially anisotropic models of the upper mantle as the one of figure 7. In general, while teleseismic body waves (which travel mostly vertically under source and receiver) are good at constraining the Earth’s lower mantle, surface wave data are the most powerful tool at our disposal to map the upper mantle: see figure 8 for a comparison between body-wave and surface-wave tomography of the Mediterranean upper mantle.

**Surface waves and azimuthal anisotropy**

We speak of “azimuthal anisotropy” when the wavespeed changes with the azimuth of the seismic ray, with respect to a fixed direction (usually the North or the East).
Figure 5: $K_{v_{SV}}$ for Rayleigh (left) and $K_{v_{SH}}$ for Love (right) waves, as a function of depth within the upper mantle under continental Europe. Different colors correspond to different periods (see legends). Only fundamental modes are shown. From Boschi et al., The Mediterranean upper mantle as seen by surface waves, 2009. See http://www.seg2.ethz.ch/boschil/publications.html
Figure 6: $K_{sv}$ for Rayleigh (left) and $K_{sh}$ for Love (right) waves, at periods of 35 s (top) and 300 s (bottom), at a depth of 70 km. from Boschi and Ekström, New images of the Earth’s upper mantle from measurements of surface-wave phase velocity anomalies, 2002. See http://www.seg2.ethz.ch/boschil/publications.html.
Smith & Dahlen\textsuperscript{4} first wrote a theoretical relation, valid in a half-space medium, between slight perturbations $\delta c$ in the phase velocity $c$ of Love and Rayleigh waves, and the azimuth $\psi$ of their direction of propagation,

$$
\frac{\delta c(r, \psi)}{c} = \epsilon_0(r) + \epsilon_1(r) \cos(2\psi) + \epsilon_2(r) \sin(2\psi) + \epsilon_3(r) \cos(4\psi) + \epsilon_4(r) \sin(4\psi),
$$

(15)

where $r$ is a 2-vector denoting position on the half space surface, and the values of $\epsilon_i$ ($i = 0, ..., 4$) naturally change also as functions of surface wave frequency $\omega$.

Starting with the work of Tanimoto and Anderson\textsuperscript{5}, equation (15) has been used to set up tomographic inverse problems, based upon the ray theory approximation, to derive global maps of the azimuthal anisotropy of Rayleigh and Love waves. In practice, you replace (15) into (1), and then parameterize each $\epsilon_i$ ($i = 0, ..., 4$) separately. Surface-wave anisotropy in the Earth has generally been found to be small, and yet significant enough to provide some insight, e.g., on the nature of mantle convection, with the fast direction of propagation a function of crystal alignment, and of the spread direction of newly formed plates at oceanic ridges.


Figure 8: A “cross-section” through the Western Mediterranean (see top map) of three different, independent images of mantle heterogeneity: a shear velocity model based on surface waves (top), a P-velocity model based on P wave travel times (middle), and a reconstruction of the temperature field based on the region’s tectonic history (bottom). The P-model has no heterogeneous crustal correction, and “smearing” takes place near the Moho. The surface wave model is very similar to the theoretical reconstruction. From Boschi, Ekström and Kustowski, *Geophys. J. Int.*, 1998. See http://www.seg2.ethz.ch/boschil/publications.html.